Two characterizations of the uniform rule based on new robustness properties*

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Abstract

We study the problem of allocating a divisible good among a group of people. Each person’s preferences are single-peaked. We consider situations in which there might be more of the resource to be assigned than was planned, or there might be less of the resource. Two robustness properties are formulated, which we call one-sided composition up and one-sided composition down. We show that only one rule satisfies irrelevance of null agent, and either the equal division lower-bound or no envy and peak only, and one of our robustness properties. This rule is the uniform rule.

JEL classification: C71; D63.

Keywords: one-sided composition up, one-sided composition down, irrelevance of null agent, uniform rule

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1 Introduction

Consider a department chair having to assign teaching duties to faculty. Each faculty member has an ideal workload. His welfare decreases as his workload moves away from his ideal amount. Each faculty member has to be assigned a non-negative workload. We look for rules allocating the total workload that satisfy certain desirable properties.

Various properties of efficiency, fairness, and robustness under strategic behavior have been analyzed for this problem (Sprumont, 1991; Ching, 1992, 1994, 2010; Thomson, 1994a, 1994b, 1995, 1997, 2011). Here, we consider the case where the total workload is not fixed. Once teaching duties have been assigned, imagine that too many students have registered for some course. In such a case, the chair has to increase the number of sections for the course. Alternatively, it may be that some course has too few students. Then, some sections have to be canceled. How should we handle these changes?

In general, when the circumstances in which a group of agents find themselves change, there might be several ways to handle the change. The document that binds the agents as a group may specify a rule to solve the decision problems they face, but which one of several perspectives may be taken in applying the rule when circumstances change is not always written down. These various perspectives may well be equally legitimate. However, they may affect the welfare of different agents differently; hurt some and benefit others. To avoid the possibility that agents be differently affected, we formulate the robustness property that no matter which perspective is taken, it should produce the same outcome.

Think of the following scenario: after a decision has been made about some first situation, opportunities expand. The first position that one may take is to declare the first situation irrelevant. Then, we would simply ignore the initial decision and handle the second situation on its own. The second position is to use the initial decision as a starting point in dealing with the second situation. Both ways of proceeding seem equally reasonable. Our robustness property is that whichever position we are taking, each agent should be indifferent between the two final decisions.

This principle has been investigated under the name of “step-by-step negotiations” (Kalai, 1977) for bargaining, and it is also reminiscent of the “path independence” axiom for choice functions (Plott, 1973). The axiom that has been examined in the context of the adjudication of conflicting claims, under the name of “composition up” (Young, 1988), can also be seen
as an expression of our robustness principle. In that context, a shrinking of opportunities has been considered too, and a dual axiom, “composition down”, formulated. It too can be understood as an expression of the principle. For the problem of allocating indivisible goods, this principle has also been studied (Abizada and Chen, 2011).

For each class of problems, a general principle needs to be adapted in order to best take into account the structure of the class. We propose two new properties that apply the robustness idea to our model. Return to the example. Our first property is related to the possibility that the total workload increases, that is, when new sections have to be offered. Two possible ways of proceeding are (i) to ignore the choice made initially, and to reapply the rule to the new problem, the one with the larger workload; (ii) to use the initial choice as point of departure. Since each faculty member keeps his initial workload, his preferences in the situation of allocating the additional sections are no longer the same as before. Thus, we should revise the faculty members’ preferences. We allocate the additional sections using these revised preferences. The final workload assigned to each faculty member is obtained by augmenting his initial workload by the additional workload computed in this manner. Each of these procedures provides a plausible way of handling the change in the total workload. However, they may affect the welfare of different faculty members differently. If that is so, depending on which procedure is chosen, some faculty members may object that the other one was not chosen. In order to avoid such situations, we require the final assignments to be the same, independently of which procedure is followed.

The second property pertains to the opposite change in the total workload, that is, when some courses have to be canceled. Two possible ways of proceeding are (i) once again, to ignore the choice made initially and to reapply the rule to the new problem, the one with the smaller workload; (ii) to use the initial choice as point of departure. Similarly, since each faculty member keeps his initial workload, we should revise his preferences in the new situation. We allocate the cancelled courses by treating them as an “endowment” (changing the sign from negative to positive), using the revised preferences. The final workload assigned to each faculty member is obtained by subtracting the amount computed in this manner from his initial workload. Again, we require the final assignments to be the same.

We consider two efficiency properties: first, there should be no assignment at which at least one faculty member is better off and no faculty member is worse off (efficiency); second,

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1 Young uses the term “composition”. The name “composition up” is proposed by Thomson (2003).

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the assignment should not be affected by the departure of faculty members who ideally prefer not to teach (*irrelevance of null agents*). We also consider two standard punctual fairness properties: first, no faculty member should prefer someone else’s assignment to his own (*no envy*); second, each faculty member should find his or her assignment at least as desirable as an equal share of the total workload (*the equal division lower-bound*).

Herrero and Villar (2000) propose a property that can be seen as another expression of the robustness principle. They consider situations in which the endowment increases. Theirs was the first paper to study the robustness principle for this model. However, this property is incompatible with *efficiency* and either one of the fairness properties. We define a similar robustness property which pertains to situations in which the endowment decreases. Again, this property is incompatible with *efficiency* and either one of our fairness properties.

Incompatibility occurs when initially there is too much (the sum of peaks is smaller than endowment) and the endowment increases, or when initially there is not enough (the sum of peaks is larger than endowment) and the endowment decreases. We propose two more reasonable robustness properties, which are conditional upon the endowment remaining on the same side of the sum of the peaks. This kind of conditioning is introduced in this model in Thomson (1994a). The first property applies to the case of “there is still not enough after an increase”, and the second property applies to the case of “there is still too much after a decrease”. These are the central notions of our paper.

Our first main result is that there is a unique rule satisfying *irrelevance of null agents*, the *equal division lower-bound*, and the two robustness properties. It is the rule known as the “uniform rule”, which is the most central rule in the literature. We also show that the uniform rule is the only rule satisfying a weaker version of *irrelevance of null agents, no envy, peak only*, and the two robustness properties.

The uniform rule has already come out of a number of axiomatic studies of this class of problems. Our study confirms its importance.

The remainder of the paper is organized as follows. In Section 2 we define the model. In Section 3 we state the axioms and define the rules. In Section 4 we provide our main results and some discussions.
2 Model

There is a finite set of potential agents \( \mathcal{N} = \{1, 2, \cdots, n\} \). Let \( N \subseteq \mathcal{N} \). Each agent \( i \in N \) has a continuous preference relation \( R_i \) defined over \( \mathbb{R} \). Let \( P_i \) be the strict preference relation associated with \( R_i \), and \( I_i \) the indifference relation. The preference relation \( R_i \) is single-peaked: there is a number \( p(R_i) \in \mathbb{R} \) such that for each pair \( x_i, y_i \in \mathbb{R} \), if \( y_i < x_i \leq p(R_i) \) or \( p(R_i) < x_i < y_i \), then \( x_i P_i y_i \). The number \( p(R_i) \) is the peak of \( R_i \).\(^3\)

\(^2\)Most of our analysis is for a fixed population. However, one mild property involves variable populations.

\(^3\)Here we allow negative peaks. Consider the example in the introduction. For each faculty member, there is a minimum of 10 hours teaching per week required by the university. This minimum requirement is the origin in our model. Some faculty member may prefer to teach only 8 hours per week. In our model, it means that his peak is at -2.

Let \( \mathcal{R} \) be the set of all such preference relations. Let \( R \equiv (R_i)_{i \in N} \in \mathcal{R}^N \) be the preference profile of \( N \). There is an endowment \( M \in \mathbb{R}_+ \) of an infinitely divisible commodity that has to be fully distributed among the agents.

A problem with agent set \( N \) is a pair \((R, M) \in \mathcal{R}^N \times \mathbb{R}_+ \). Let \( \Pi^N \) be the set of problems with agent set \( N \). An assignment for \((R, M)\) is a list \( x = (x_i)_{i \in N} \in \mathbb{R}_+^N \) such that \( \sum_{i \in N} x_i = M \). Let \( X(R, M) \) be the set of assignments for \((R, M)\). A rule associates with each problem an assignment for it. Formally, it is a mapping \( \varphi : \bigcup_{N \subseteq \mathcal{N}} \Pi^N \to \bigcup_{N \subseteq \mathcal{N}} \mathbb{R}_+^N \) such that for each \( N \subseteq \mathcal{N} \) and each \((R, M) \in \Pi^N \), \( \varphi(R, M) \in X(R, M) \).

3 Rules and axioms

First, we define various rules that have played an important role in the literature. They will also help us understand the strength of our axioms and establish their independence in our characterizations.

First is the rule that assigns each agent an equal share of the endowment.

**Equal division rule, \( EDiv \):** For each \( N \subseteq \mathcal{N} \), each \((R, M) \in \Pi^N \), and each \( i \in N \)

\[
EDiv_i(R, M) = \frac{M}{|N|}.
\]
The second rule is the expression for this model of an idea that has played a central role in the theory of fairness, that of proportionality. It assigns each agent an equal portion of his most preferred non-negative amount.

**Proportional rule, Prop**: For each \( N \subseteq \mathcal{N} \), each \((R, M) \in \Pi^N\), and each \( i \in N \)

\[
Prop_i(R, M) = \begin{cases}
\bar{p}(R_i)M & \text{if } \sum_{j \in N} \bar{p}(R_j) > 0, \\
\frac{M}{|N|} & \text{otherwise}.
\end{cases}
\]

The next rule evaluates sacrifices made by agents at an allocation by distances to peaks and it seeks to equate sacrifices, adjustment being made if needed to take into account our non-negativity constraint.

**Equal distance rule, EDist**: For each \( N \subseteq \mathcal{N} \), each \((R, M) \in \Pi^N\), and each \( i \in N \)

\[
EDist_i(R, M) = \max\{0, \bar{p}(R_i) + \lambda\}
\]

where \( \lambda \in \mathbb{R} \) solves \( \sum_{j \in N} EDist_j(R, M) = M \).

The last rule is arguably the most central rule in the literature. It gives agents an equal opportunity by choosing a closed interval, and assigning each one his most preferred point in the interval. When there is not enough, the left endpoint of the interval is 0. When there is too much, the right endpoint of the interval is \( M \).

**Uniform rule, U**: For each \( N \subseteq \mathcal{N} \), each \((R, M) \in \Pi^N\), and each \( i \in N \),

\[
U_i(R, M) = \begin{cases}
\min\{\bar{p}(R_i), \lambda\} & \text{if } \sum_{j \in N} \bar{p}(R_j) \geq M, \\
\max\{\bar{p}(R_i), \lambda\} & \text{otherwise}.
\end{cases}
\]

where \( \lambda \in \mathbb{R}^+ \) solves \( \sum_{j \in N} U_j(R, M) = M. \)

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4It is defined in Bénassy (1982). The first axiomatic characterization of it is provided by Sprumont (1991).
Next, we introduce the axioms, beginning with punctual fairness notions. Let $\varphi$ be a rule.

- In discussing fairness, equal division is frequently thought of as a natural reference point. Equal division is typically not efficient, but if we think of equal division as specifying ownership rights, an appealing fairness property is that each agent should find his assignment at least as desirable as an equal share of the endowment. It is satisfied by the equal division rule and the uniform rule, but is violated by the other two rules that we defined.

**Equal division lower-bound:** For each $N \subseteq \mathcal{N}$, each $(R, M) \in \Pi^N$, and each $i \in N$, $\varphi_i(R, M) R_i \frac{M}{|N|}$.

- Another way of assessing the fairness of an assignment is to let each agent compare his assignment to everybody else’s assignment. We require that each agent should find his assignment at least as desirable as each other agent’s assignment (Foley, 1967). It is satisfied by the equal division rule and the uniform rule, but is violated by the other two rules that we defined.

**No envy:** For each $N \subseteq \mathcal{N}$, each $(R, M) \in \Pi^N$, and each pair $i, j \in N$, $\varphi_i(R, M) R_i \varphi_j(R, M)$.\(^5\)

Next, we discuss efficiency notions.

- The standard property here is that there should be no assignment at which each agent is at least as well off, and some agent is better off. It is violated by the equal division rule, but is satisfied by the other three.

**Efficiency:** For each $N \subseteq \mathcal{N}$ and each $(R, M) \in \Pi^N$, there is no $x \in X(R, M)$ at which for each $i \in N$, $x_i R_i \varphi_i(R, M)$, and for some $j \in N$, $x_j P_j \varphi_j(R, M)$.

- The next property is that when there is excess demand, removing the agents with peaks

\(^5\)There is no logical relation between our two fairness notions: the *equal division lower-bound* and *no envy* (Thomson, 1994b, 1995).
at zero should not affect the other agents’ assignments. It is violated by the equal division rule, but is satisfied by the other three.

**Irrelevance of null agents:** For each $N \subseteq \mathcal{N}$ and each $(R, M) \in \Pi^N$ with $N^0 \equiv \{i \in N : p(R_i) = 0\}$, if $\sum_{j \in N} \bar{p}(R_j) \geq M$, then for each $j \in N \setminus N^0$, $\varphi_j(R, M) = \varphi_j(R_{N \setminus N^0}, M)$.

- A weaker property is that when there is excess demand, each agent with a peak at zero should be assigned nothing. It is still violated by the equal division rule.

**Weak irrelevance of null agents:** For each $N \subseteq \mathcal{N}$ and each $(R, M) \in \Pi^N$, if $\sum_{j \in N} \bar{p}(R_j) \geq M$, then for each $i \in N$ such that $p(R_i) = 0$, $\varphi_i(R, M) = 0$.

Note that both efficiency and irrelevance of null agents individually imply weak irrelevance of null agents. There is no logical relationship between these two.

- The assignment should only depend on the peak profile. This axiom may not be as appealing as the above ones, but it is satisfied by all the four rules. It has also played a central role in many studies of the problem.

**Peak only:** For each $N \subseteq \mathcal{N}$, each $(R, M) \in \Pi^N$, and each $R' \in \mathcal{R}^N$, if for each $i \in N$, $p(R'_i) = p(R_i)$, then $\varphi(R', M) = \varphi(R, M)$.

Finally we discuss several robustness properties pertaining to changes in the endowment. These are the ones that are the focus of our study.

- The first robustness property is introduced by Herrero and Villar (2000). Suppose that after an assignment is chosen, the endowment increases. As informally discussed in the introduction, two perspectives can be taken in deciding what to do in the new situation. The

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Herrero and Villar (2000) define a notion called “dummy”. It requires that once the assignment is selected, if a group of agents each of whom receives zero leave, the assignment selected by the rule for the remaining agents should be the same. This property, together with weak irrelevance of null agents, imply irrelevance of null agents.
first one is to ignore the initial assignment, and to reapply the rule to the new problem. The second one is to use the initial assignment as point of departure: we let each agent \( i \in N \) keep his initial assignment \( x_i \); and we only distribute the incremental amount. We revise each agent’s preferences by the amount he was initially assigned. Then, we allocate the incremental amount by using the revised preferences. The assignment should be independent of the chosen perspective.

To formally definite the property, we need additional notations:

Let \( (R,M) \in \Pi^N \) and \( x \in X(R,M) \). For each \( i \in N \), let \( R_i^{-x_i} \in R \) be the relation defined by shifting the preference relation \( R_i \) by the amount \( x_i \). Formally, for each pair \( a,b \in \mathbb{R}, (a - x_i) R_i^{-x_i} (b - x_i) \) if and only if \( a R_i b \). Let \( R^{-x} \equiv (R_i^{-x_i})_{i \in N} \).

Composition up: For each \( N \subseteq N \), each \( (R,M) \in \Pi^N \), and each \( M' \in \mathbb{R}_+ \) with \( M' > M \),

\[
\varphi(R,M') = \varphi(R,M) + \varphi(R^{-\varphi(R,M)}, M' - M).
\]

Composition up is satisfied by the equal division rule and by the equal distance rule. But it is violated by the other rules introduced above. In fact, it turns out to be incompatible with efficiency and either one of our fairness properties, no-envy and the equal division lower-bound. For the next two propositions, the very weak form of efficiency introduced above is imposed instead. We still get an incompatibility.

**Proposition 1** No rule satisfies weak irrelevance of null agents, the equal division lower-bound, and composition up.
Proof. Suppose by contradiction that there exists a rule $\varphi$ satisfying the three axioms of the proposition.

Let $N = \{1, 2\}$ and $R \in \mathcal{R}^N$ be such that $p(R_1) = 0$ and $p(R_2) = 2$. Let $M = 2$ and $M' = 4$. Since $p(R_1) + p(R_2) = 2 = M$, by weak irrelevance of null agents, $\varphi_1(R, M) = 0$. Thus, $\varphi_2(R, M) = 2$. Since $p(R) = (0, 2)$ and $\varphi(R, M) = (0, 2)$, we have $p(R^{-\varphi(R,M)}) = (0, 0)$. By the equal division lower bound, $\varphi(R^{-\varphi(R,M)}, M' - M) = (1, 1)$. By composition up, $\varphi(R, M') = \varphi(R, M) + \varphi(R^{-\varphi(R,M)}, M' - M) = (1, 3)$. Since $\frac{M'}{|N|} = 2$ and $p(R_2) = 2$, agent 2 prefers $\frac{M'}{|N|}$ to $\varphi_2(R, M')$. This violates the equal division lower bound.

Proposition 2 No rule satisfies weak irrelevance of null agents, no envy, and composition up.

Proof. Suppose by contradiction that there exists a rule $\varphi$ satisfying the three axioms of the proposition.

Let $N = \{1, 2\}$ and $R \in \mathcal{R}^N$ be such that $p(R_1) = 0$, $p(R_2) = 2$, and $1 P_2 3$. Let $M = 2$ and $M' = 4$. Since $p(R_1) + p(R_2) = 2 = M$, by weak irrelevance of null agents, $\varphi_1(R, M) = 0$. Thus, $\varphi_2(R, M) = 2$. Since $p(R) = (0, 2)$ and $\varphi(R, M) = (0, 2)$, we have $p(R^{-\varphi(R,M)}) = (0, 0)$. By no envy, $\varphi(R^{-\varphi(R,M)}, M' - M) = (1, 1)$. By composition up, $\varphi(R, M') = \varphi(R, M) + \varphi(R^{-\varphi(R,M)}, M' - M) = (1, 3)$. Since $1 P_2 3$, agent 2 envies agent 1 at $\varphi(R, M')$. This violates no envy.

Note that in the proofs of Propositions 1 and 2, stating the incompatibility of efficiency, either one of our fairness properties, and composition up, a difficulty occurred when initially there is not enough of the good and after the endowment increased it becomes too much. Therefore, we focus on endowment increases, conditional upon the endowment remaining on the same side of the sum of the peaks.\footnote{It is not surprising that this one-sided conditioning is relevant to the possibility of rules satisfying a property. In fact, it has appeared in several previous studies (Thomson, 1994a, 1995, 1997).} We define a weaker notion of composition up in the following way: when there is not enough, we only consider increases in the endowment such that the direction of the inequality between the sum of the peaks and the endowment doesn’t change. The following property is one of our two central notions.
**One-sided composition up:** For each $N \subseteq \mathcal{N}$, each $(R, M) \in \Pi^N$, and each $M' \in \mathbb{R}_+$,

$$
\sum_{i \in N} \bar{p}(R_i) \geq M' > M \implies \phi(R, M') = \phi(R, M) + \phi(R^{-\phi(R, M)}, M' - M),
$$

- The second robustness property is related to decreases in endowment. Suppose that after an assignment is chosen, the endowment decreases. Once again, two perspectives can be taken. The first one is to ignore the initial assignment, and to reapply the rule to the new problem. The second one is to use the initial assignment as point of departure: we let each agent $i \in \mathcal{N}$ to keep his initial assignment $x_i$, and distribute the shortfall. We revise each agent’s preferences by the amount he was initially assigned. Since the shortfall is negative and feasibility requires that each agent should receive a nonnegative amount, we need further revision. By changing the sign of the shortfall, we treat it as an endowment. To be consistent, we also need to revise preferences by using the operator “$\text{sym}$”. Then, we apply the rule to this problem and subtract the assignment we obtain from the initial one. Now, we write that the assignment should be independent of the chosen perspective.

To formally define our second robustness property, we introduce some more notations:

Let $(R, M) \in \Pi^N$. For each $i \in \mathcal{N}$, let $\text{sym}(R_i) \in \mathcal{R}$ be the relation defined as follows: for each pair $a, b \in \mathbb{R}$, $(-a) \text{sym}(R_i)(-b)$ if and only if $a R_i b$. Let $\text{sym}(R) \equiv (\text{sym}(R_i))_{i \in \mathcal{N}}$.

\[ R_i \quad \xrightarrow{\text{sym}} \quad \text{sym}(R_i) \]

Figure 2: **Symmetry operation on preferences.** In (a), the preference relation of agent $i$, $R_i$, is depicted. We define relation $\text{sym}(R_i)$ for agent $i$ by taking symmetric image of his preference relation $R_i$ with respect to the origin, as depicted in (b).

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8Although the term “one-sidedness” is consistent with the terminology in Thomson (1994a, 1995, 1997), our property is even weaker since we only consider changes that occur on the excess demand side.
**Composition down:** For each $N \subseteq \mathcal{N}$, each $(R, M) \in \Pi^N$, and each $M' \in \mathbb{R}_+$ with $M' < M$,

$$\varphi(R, M') = \varphi(R, M) - \varphi(\text{sym}(R^{-\varphi(R,M)}), M - M').$$

Among the rules we introduced, *composition down* is only satisfied by the equal division rule.

In fact, the property is incompatible with either one of our fairness properties, and a very mild efficiency property.

**Proposition 3** No rule satisfies weak irrelevance of null agents, the equal division lower-bound, and composition down.

**Proof.** Suppose by contradiction that there is a rule $\varphi$ satisfying the three axioms of the proposition.

Let $N = \{1, 2\}$ and $R \in \mathcal{R}^N$ be such that $p(R_1) = 0$ and $p(R_2) = 2$. Let $M = 2$ and $M' = 1$. Since $p(R_1) + p(R_2) = 2 = M$, by weak irrelevance of null agents, $\varphi_1(R, M) = 0$. Thus, $\varphi_2(R, M) = 2$. Since $p(R) = (0, 2)$ and $\varphi(R, M) = (0, 2)$, we have $p(\text{sym}(R^{-\varphi(R,M)})) = (0, 0)$. By the equal division lower bound, $\varphi(\text{sym}(R^{-\varphi(R,M)}), M' - M) = (\frac{1}{2}, \frac{1}{2})$. By composition down, $\varphi(R, M') = \varphi(R, M) - \varphi(\text{sym}(R^{-\varphi(R,M)}), M' - M) = (0, 2) - (\frac{1}{2}, \frac{1}{2}) = (\frac{1}{2}, \frac{3}{2})$. This violates non-negativity of assignments.

**Proposition 4** No rule satisfies weak irrelevance of null agents, no envy, and composition down.

**Proof.** The proof is similar to that of Proposition 3 and we omit it.

Note that as before, in the proofs of Propositions 3 and 4, stating the incompatibility of efficiency, either one of the fairness properties, and composition down, a difficulty occurred when initially there is too much of the good and after the endowment decreased it becomes not enough. Therefore, we focus on endowment decreases, conditional upon the endowment remaining on the same side of the sum of the peaks. We define our weaker notion of composition down.
sition down in the following way: when there is too much, we only consider decreases in the endowment such that the direction of the inequality between the sum of the peaks and the endowment doesn’t change. The following property is the second one of our central notions.

One-sided\textsuperscript{9} composition down: For each $N \subseteq \mathcal{N}$, each $(R, M) \in \Pi^N$, and each $M' \in \mathbb{R}_+$,

$$\sum_{i \in N} \bar{p}(R_i) < M' < M \implies \varphi(R, M') = \varphi(R, M) - \varphi(\text{sym}(R^{-\varphi(R,M)}), M - M').$$

Our two one-sided versions of composition properties, seem to be the most natural way of weakening our composition properties in the sense that we impose them only when “desirable” changes in the endowment occur, i.e. when there is not enough and the endowment increases, and when there is too much and the endowment decreases. The resulting properties turn out to be satisfied by all four rules that we introduced.

4 Main Results

Although our robustness properties are weak, when we impose them together with fairness and a very mild efficiency property, we obtain one single rule. This rule is the uniform rule. Note that efficiency itself is not imposed as one of our axioms. Yet, the uniform rule is efficient. Kesten (2006), Ching (2010), and Ehlers (2011) provide characterizations of the uniform rule, without imposing efficiency.

Theorem 1 The uniform rule is the only rule satisfying irrelevance of null agents, the equal division lower-bound, one-sided composition up, and one-sided composition down.

Proof. It is easy to verify that the uniform rule satisfies all the axioms. Conversely, let $\varphi$ be a rule satisfying these axioms. Let $N \subseteq \mathcal{N}$, $(R, M) \in \Pi^N$, and $n \equiv |N|$. We distinguish two cases:

\textsuperscript{9}As it is with one-sided composition up, the term “one-sidedness” is consistent with the terminology in Thomson (1994a, 1995, 1997). Our property is even weaker since we only consider changes that occur on the excess supply side.
Case 1. $M \in \left[0, \sum_{i \in N} \bar{p}(R_i) \right]$.  

Let $N^0 \equiv \{i \in N : p(R_i) \leq 0 \}$, and $n^0 \equiv |N^0|$  
Let $p^1 \equiv \min_{i \in N \setminus N^0} p(R_i)$, $N^1 \equiv \{i \in N : p(R_i) = p^1 \}$, and $n^1 \equiv |N^1|$.

**Subcase 1.1.** $M \in \left[0, (n-n^0)p^1 \right]$.  
By irrelevance of null agents,  
$$\sum_{j \in N \setminus N^0} \varphi_j(R, M) = \sum_{j \in N \setminus N^0} \varphi_j(R_{N \setminus N^0}, M) = M.$$  
Thus, for each $i \in N^0$, $\varphi_i(R, M) = 0$.  
By the equal division lower-bound, for each $i \in N \setminus N^0$, $\varphi_i(R_{N \setminus N^0}, M) R_i \frac{M}{n-n^0}$.  
Since $\frac{M}{n-n^0} \leq p^1$, for each $i \in N \setminus N^0$, $\varphi_i(R_{N \setminus N^0}, M) \geq \frac{M}{n-n^0}$. Thus, for each $i \in N \setminus N^0$, $\varphi_i(R_{N \setminus N^0}, M) = \frac{M}{n-n^0}$.  
By irrelevance of null agents, $\varphi_{N \setminus N^0}(R, M) = \varphi(R_{N \setminus N^0}, M)$. Thus, for each $i \in N \setminus N^0$, $\varphi_i(R, M) = \frac{M}{n-n^0}$.

Let $p^2 \equiv \min_{i \in N \setminus (N^0 \cup N^1)} p(R_i)$, $N^2 \equiv \{i \in N : p(R_i) = p^2 \}$, and $n^2 \equiv |N^2|$.

**Subcase 1.2.** $M \in ((n-n^0)p^1, n^1p^1 + (n-n^0-n^1)p^2]$.  
Let $M' \equiv (n-n^0)p^1$. By one-sided composition up, $\varphi(R, M) = \varphi(R, M') + \varphi(R^{-\varphi(R,M')} , M - M')$.  
By Subcase 1.1, for each $i \in N^0$, $\varphi_i(R, M') = 0$, and for each $i \in N^1$, $\varphi_i(R, M') = \frac{M'}{n-n^0} = p^1$. Since for each $i \in N^0 \cup N^1$, $p(R_i^{-\varphi(R,M')}) = 0$, by irrelevance of null agents,  
$$\sum_{j \in N \setminus (N^0 \cup N^1)} \varphi_j(R^{-\varphi(R,M')}, M-M') = \sum_{j \in N \setminus (N^0 \cup N^1)} \varphi_j(R^{-\varphi(R,M')}, M-M')|_{N \setminus (N^0 \cup N^1)} = M-M'.$$  
Thus, for each $i \in N^0 \cup N^1$, $\varphi_i(R^{-\varphi(R,M')}, M-M') = 0$. Thus, for each $i \in N^0$, $\varphi_i(R, M) = 0 + 0 = 0$, and for each $i \in N^1$, $\varphi_i(R, M) = p^1 + 0 = p^1$.  
By Subcase 1.1, for each $i \in N \setminus (N^0 \cup N^1)$, $\varphi_i(R, M') = p^1$. By the equal division lower-bound, for each $i \in N \setminus (N^0 \cup N^1)$, $\varphi_i(R^{-\varphi(R,M')}|_{N \setminus (N^0 \cup N^1)}, M-M') R_i \frac{M-M'}{n-n^0-n^1}$. Since $\frac{M-M'}{n-n^0-n^1} \leq p^2 - p^1$, for each $i \in N \setminus (N^0 \cup N^1)$, $\varphi_i(R^{-\varphi(R,M')}|_{N \setminus (N^0 \cup N^1)}, M-M') \geq \frac{M-M'}{n-n^0-n^1}$.  
Thus, for each $i \in N \setminus (N^0 \cup N^1)$, $\varphi_i(R^{-\varphi(R,M')}|_{N \setminus (N^0 \cup N^1)}, M-M') = \frac{M-M'}{n-n^0-n^1}$. By irrelevance of null agents, $\varphi_{N \setminus (N^0 \cup N^1)}(R^{-\varphi(R,M')}, M-M') = \varphi(R^{-\varphi(R,M')}|_{N \setminus (N^0 \cup N^1)}, M-M')$. Thus, for each $i \in N \setminus (N^0 \cup N^1)$, $\varphi_i(R, M) = p^1 + \frac{M-M'}{n-n^0-n^1} = \frac{(n-n^0-n^1)p^1 + M-(n-n^0)n^1}{n-n^0-n^1} = \frac{M-n^1p^1}{n-n^0-n^1}$.
Let $p^3 \equiv \min_{i \in N \setminus (\bar{N} \cup N^1 \cup N^2)} p(R_i)$, $N^3 \equiv \{i \in N : p(R_i) = p^3\}$, and $n^3 \equiv |N^3|$.

**Subcase 1.3.** $M \in (n^1 p^1 + (n - n^0 - n^1)p^2, n^1 p^1 + n^2 p^2 + (n - n^0 - n^1 - n^2)p^3]$.

Repeating the argument, we obtain that if $M \in [0, \sum_{i \in N} \bar{p}(R_i)]$, then $\varphi(R, M) = U(R, M).

**Case 2.** $M \in (\sum_{i \in N} \bar{p}(R_i), \infty)$.

Let $p^k \equiv \max_{i \in N} p(R_i)$, $N^k \equiv \{i \in N : p(R_i) = p^k\}$, and $n^k \equiv |N^k|$.  

**Subcase 2.1.** $M \in [np^k, \infty)$.

By the equal division lower-bound, for each $i \in N$, $\varphi_i(R, M) \leq M_{\frac{M}{n}}$. Since $\frac{M}{n} \geq p^k = \max_{i \in N} p(R_i)$, for each $i \in N$, $\varphi_i(R, M) \leq \frac{M}{n}$. Since $\sum_{i \in N} \varphi_i(R, M) = M$, for each $i \in N$, $\varphi_i(R, M) = \frac{M}{n}$.

**Subcase 2.2.** $M \in (\sum_{i \in N} \bar{p}(R_i), np^k)$. Let $M' = np^k$. By one-sided composition down, $\varphi(R, M) = \varphi(R, M') - \varphi(\text{sym}(R^{-\varphi(R,M')}), M' - M)$. 

By Subcase 2.1, for each $i \in N$, $\varphi_i(R, M') = p^k$. Let $\bar{R} \equiv \text{sym}(R^{-\varphi(R,M')})$. Then, for each $i \in N$, $p(\bar{R}_i) = -[p(R_i) - \varphi_i(R, M')] = p^k - p(R_i)$. Since $\sum_{i \in N} \bar{p}(R_i) < M$, we have

$$\sum_{i \in N} \bar{p}(R_i) = \sum_{i \in N} p(R_i) = np^k - \sum_{i \in N} p(R_i) \geq np^k - \sum_{i \in N} \bar{p}(R_i) > M' - M.$$ 

By Case 1, for each $i \in N$, $\varphi_i(\bar{R}, M' - M) = \min\{p(\bar{R}_i), \lambda\}$, where $\lambda$ solves $\sum_{j \in N} \varphi_j(\bar{R}, M' - M) = M' - M$. Since for each $i \in N$, $p(\bar{R}_i) = p^k - p(R_i)$, we have $\min\{p(\bar{R}_i), \lambda\} = p^k - \max\{p(R_i), p^k - \lambda\}$. Thus, for each $i \in N$, $\varphi_i(R, M) = p^k - [p^k - \max\{p(R_i), p^k - \lambda\}] = \max\{p(R_i), p^k - \lambda\}$. Let $\lambda' \equiv p^k - \lambda$. Since $\sum_{j \in N} \min\{p(R_j), \lambda\} = M' - M$, we have $\sum_{j \in N} \max\{p(R_j), \lambda'\} = \sum_{j \in N} [p^k - \min\{p(R_j), \lambda\}] = np^k - (M' - M) = M$.

Thus, for each $i \in N$, $\varphi_i(R, M) = \max\{p(R_i), \lambda'\}$, where $\lambda'$ solves $\sum_{j \in N} \varphi_j(R, M) = M$. Thus, if $M \in (\sum_{i \in N} \bar{p}(R_i), np^k)$, then $\varphi(R, M) = U(R, M)$. 

On the independence of the axioms in Theorem 1

(1) The equal division rule satisfies all the axioms of Theorem 1 except for irrelevance of null agents.

(2) The proportional rule satisfies all the axioms of Theorem 1 except for the equal-division lower bound.

(3) Consider the “mixed” rule defined as follows: for each $N \subseteq \mathcal{N}$, each $(R, M) \in \Pi^N$, and each $i \in N$,

$$Mix_i(R, M) = \begin{cases} U_i(R, M) & \text{if } \sum_{j \in N} p(R_j) \geq M, \\ EDiv_i(R, M) & \text{otherwise.} \end{cases}$$

The mixed rule satisfies all the axioms of Theorem 1 except for one-sided composition down.

Example 1. Let $N = \{1, 2\}$ and $R \in \mathcal{R}^N$ be such that $p(R_1) = 2$ and $p(R_2) = 6$. Let $M = 12$ and $M' = 10$.

Since $2 + 6 < 12$, we have $Mix(R, M) = EDiv(R, M) = (6, 6)$. Since $2 + 6 < 10$, we have $Mix(R, M') = EDiv(R, M'') = (5, 5)$.

Let $\bar{R} \equiv \text{sym}(R^{-Mix(R,M)})$. Then, $p(\bar{R}_1) = 4$ and $p(\bar{R}_2) = 0$. Since $4 + 0 > 12 - 10$, we have $Mix(\bar{R}, M' - M) = U(\bar{R}, M' - M) = (2, 0)$.

Then, $Mix(R, M) - Mix(\bar{R}, M' - M) = (4, 6) \neq (5, 5) = Mix(R, M')$. This is in violation of one-sided composition down.

(4) Consider the “reverse mixed” rule defined as follows: for each $N \subseteq \mathcal{N}$, each $(R, M) \in \Pi^N$, and each $i \in N$,

$$RMix_i(R, M) = \begin{cases} U_i(R, M) & \text{if } \sum_{j \in N} p(R_j) \leq M, \\ \frac{M}{|\{j \in N: p(R_j) > 0\}|} & \text{if } \sum_{j \in N} p(R_j) > M \text{ and } p(R_i) > 0, \\ 0 & \text{if } \sum_{j \in N} p(R_j) > M \text{ and } p(R_i) = 0, \end{cases}$$

The reverse mixed rule satisfies all the axioms of Theorem 1 except for one-sided composition up.

\footnote{A similar rule where $EDiv$ is replaced with $EDist$ is defined in Herrero and Villar (1998).}
Example 2. Let $N = \{1, 2, 3\}$ and $R \in \mathcal{R}^N$ be such that $p(R_1) = 2$, $p(R_2) = 4$ and $p(R_3) = 4$. Let $M = 6$ and $M' = 9$.

Since $2 + 4 + 4 > 6$, we have $RMix(R, M) = (2, 2, 2)$. Since $2 + 4 + 4 > 9$, we have $RMix(R, M') = (3, 3, 3)$.

Let $\bar{R} \equiv \text{sym}(R^{-Mix(R, M)})$. Then, $p(R_1) = 0$, $p(R_2) = 2$ and $p(R_3) = 2$. Since $2 + 2 > 3$ and since $p(R_1) = 0$, we have $RMix(\bar{R}, M' - M) = (0, 1.5, 1.5)$.

Then, $RMix(R, M) + RMix(\bar{R}, M' - M) = (2, 3.5, 3.5) \neq (3, 3, 3) = RMix(R, M')$. This is in violation of one-sided composition up.

In our next Theorem, we weaken irrelevance of null agents to weak irrelevance of null agents, replace the equal division lower-bound with no envy, and add peak only. We obtain another characterization of the uniform rule, once again, without imposing efficiency.

Theorem 2 The uniform rule is the only rule satisfying weak irrelevance of null agents, no envy, peak only, one-sided composition up, and one-sided composition down.

Proof. It is easy to verify that the uniform rule satisfies all the axioms. Conversely, let $\varphi$ be a rule satisfying these axioms. Let $N \subseteq \mathcal{N}$, $(R, M) \in \Pi^N$, and $n \equiv |N|$. We distinguish several cases:

Case 1. $M \in [0, \sum_{i \in N} \bar{p}(R_i)]$.

Let $N^0 \equiv \{i \in N : p(R_i) \leq 0\}$, and $n^0 \equiv |N^0|$.

Let $p^1 \equiv \min_{i \in N \setminus N^0} p(R_i)$, $N^1 \equiv \{i \in N : p(R_i) = p^1\}$, and $n^1 \equiv |N^1|$.

Subcase 1.1. $M \in [0, (n - n^0)p^1)$.

By weak irrelevance of null agents, for each $i \in N^0$, $\varphi_i(R, M) = 0$.

We show that if $M \in [0, p^1]$, then for each $i \in N \setminus N^0$, $\varphi_i(R, M) = \frac{M}{n - n^0}$.

Since $M \leq p^1$, for each $i \in N \setminus N^0$, $\varphi_i(R, M) \leq p^1$. Suppose that there is $j \in N \setminus N^0$ such that $\varphi_j(R, M) > \frac{M}{n - n^0}$. Then, there is $k \in N \setminus \{N^0 \cup j\}$ such that $\varphi_k(R, M) < \frac{M}{n - n^0}$. Then,
We show that if $M \in (p^1, p^1 + \frac{n-n^0-1}{n-n^0}p^1]$, then for each $i \in N \setminus N^0$, $\varphi_i(R, M) = \frac{M}{n-n^0}$.

Let $M' \equiv p^1$. By one-sided composition up, $\varphi(R, M) = \varphi(R, M') + \varphi(R^{-\varphi(R, M')}, M - M')$. By the above result, for each $i \in N \setminus N^0$, $\varphi_i(R, M') = \frac{M'}{n-n^0} = \frac{p^1}{n-n^0}$. Then, $\min_{i \in N \setminus N^0} p(R_i^{-\varphi_i(R, M')}) = \frac{n-n^0-1}{n-n^0}p^1 \geq M - M'$. By the above result, for each $i \in N \setminus N^0$, $\varphi_i(R^{-\varphi(R, M')}, M - M') = M - M'$. Thus, for each $i \in N \setminus N^0$, $\varphi_i(R, M) = \frac{M'}{n-n^0} + \frac{M-M'}{n-n^0} = \frac{M}{n-n^0}$.

\ldots

In general, for each $t \in \mathbb{N}$, if $M \in \left(\sum_{k=0}^{t-2} \left(\frac{n-n^0-1}{n-n^0}\right)^k p^1, \sum_{k=0}^{t-1} \left(\frac{n-n^0-1}{n-n^0}\right)^k p^1\right]$, then for each $i \in N \setminus N^0$, $\varphi_i(R, M) = \frac{M}{n-n^0}$.

Since $\sum_{k=0}^{\infty} \left(\frac{n-n^0-1}{n-n^0}\right)^k p^1 = (n-n^0)p^1$, if $M \in [0, (n-n^0)p^1)$, then for each $i \in N \setminus N^0$, $\varphi_i(R, M) = \frac{M}{n-n^0}$.

**Subcase 1.2.** $M = (n-n^0)p^1$.

By weak irrelevance of null agents, for each $i \in N^0$, $\varphi_i(R, M) = 0$.

By one-sided composition up, for each $\bar{N} \subseteq N$, each $(\bar{R}, \bar{M}) \in \Pi^{\bar{N}}$, and each $\bar{M}' \in \mathbb{R}_+$, if $\sum_{i \in N} p(R_i) \geq \bar{M}' > \bar{M}$, then for each $i \in \bar{N}$, $\varphi_i(\bar{R}, \bar{M}') \geq \varphi_i(\bar{R}, \bar{M})$. Thus, for each $M' < (n-n^0)p^1$ and each $i \in N \setminus N^0$, $\varphi_i(R, M) \geq \varphi_i(R, M') = \frac{M'}{n-n^0}$.

Since $\sum_{i \in N \setminus N^0} \varphi_i(R, M) = (n-n^0)p^1$, for each $i \in N \setminus N^0$, $\varphi_i(R, M) = p^1$.

Let $p^2 \equiv \min_{i \in N \setminus (N^0 \cup N^1)} p(R_i)$, $N^2 \equiv \{i \in N : p(R_i) = p^2\}$, and $n^2 \equiv |N^2|$.

**Subcase 1.3.** $M \in ((n-n^0)p^1, n^1p^1 + (n-n^0-n^1)p^2]$. 

By weak irrelevance of null agents, for each $i \in N^0$, $\varphi_i(R, M) = 0$.

Let $M' \equiv (n-n^0)p^1$. By one-sided composition up, $\varphi(R, M) = \varphi(R, M') + \varphi(R^{-\varphi(R, M')}, M - M')$. By Subcase 1.2, for each $i \in N^1$, $\varphi_i(R, M') = p^1$. Since for each $i \in N^1$, $p(R_i^{-\varphi_i(R, M')}) = 0$, by weak irrelevance of null agents, $\varphi_i(R^{-\varphi(R, M')}, M - M') = 0$. Thus, for each $i \in N^1$, $\varphi_i(R, M) = p^1 + 0 = p^1$.

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We show that if \( M \in ((n - n^0)p^1, (n - n^0)p^1 + (p^2 - p^1)] \), then for each \( i \in N \setminus (N^0 \cup N^1) \),
\[ \varphi_i(R, M) = \frac{M - n^1 p^1}{n - n^0 - n^1}. \]
Let \( M' \equiv (n - n^0)p^1 \). By one-sided composition up, \( \varphi(R, M) = \varphi(R, M') + \varphi(R - \varphi(R, M'), M - M') \). By Subcase 1.2, for each \( i \in N \setminus (N^0 \cup N^1) \),
\[ \varphi_i(R, M') = \frac{M'}{n - n^0 - n^1} = p^1. \]
Then,
\[ \min_{i \in N \setminus (N^0 \cup N^1)} p(R_i - \varphi_i(R, M')) = p^2 - p^1 \geq M - M'. \]
By Subcase 1.1, for each \( i \in N \setminus (N^0 \cup N^1) \),
\[ \varphi_i(R - \varphi_i(R, M'), M - M') = \frac{M - M'}{n - n^0 - n^1}. \]
Thus, for each \( i \in N \setminus (N^0 \cup N^1) \),
\[ \varphi_i(R, M) = p^1 + \frac{M - M'}{n - n^0 - n^1} = \frac{M - n^1 p^1}{n - n^0 - n^1}. \]

We show that if \( M \in ((n - n^0)p^1 + (p^2 - p^1), (n - n^0)p^1 + (p^2 - p^1) + (n - n^0 - n^1 - 1)(p^2 - p^1)] \),
then for each \( i \in N \setminus (N^0 \cup N^1) \),
\[ \varphi_i(R, M) = \frac{M - n^1 p^1}{n - n^0 - n^1}. \]
Let \( M' \equiv (n - n^0)p^1 + (p^2 - p^1) \). By one-sided composition up, \( \varphi(R, M) = \varphi(R, M') + \varphi(R - \varphi(R, M'), M - M') \). By the above result, for each \( i \in N \setminus (N^0 \cup N^1) \),
\[ \varphi_i(R, M') = \frac{M' - n^1 p^1}{n - n^0 - n^1}. \]
By Subcase 1.1, for each \( i \in N \setminus (N^0 \cup N^1) \),
\[ \varphi_i(R - \varphi_i(R, M'), M - M') = \frac{M - M'}{n - n^0 - n^1}. \]
Thus, for each \( i \in N \setminus (N^0 \cup N^1) \),
\[ \varphi_i(R, M) = \frac{M' - n^1 p^1}{n - n^0 - n^1} + \frac{M - M'}{n - n^0 - n^1} = \frac{M - n^1 p^1}{n - n^0 - n^1}. \]

In general, for each \( t \in \mathbb{N} \), if \( M \in ((n - n^0)p^1 + \sum_{k=0}^{i-2} \left( \frac{n - n^0 - n^1 - 1}{n - n^0 - n^1} \right)^k (p^1 - p^2), (n - n^0)p^1 + \sum_{k=0}^{i-1} \left( \frac{n - n^0 - n^1 - 1}{n - n^0 - n^1} \right)^k (p^2 - p^1)] \),
then for each \( i \in N \setminus N^0 \),
\[ \varphi_i(R, M) = \frac{M - n^1 p^1}{n - n^0 - n^1}. \]

Since \[ \sum_{k=0}^{\infty} \left( \frac{n - n^0 - n^1 - 1}{n - n^0 - n^1} \right)^k (p^2 - p^1) = (n - n^0 - n^1)(p^2 - p^1) \] and \( (n - n^0)p^1 + (n - n^0 - n^1)(p^2 - p^1) = n^1 p^1 + (n - n^0 - n^1)p^2 \),
then for each \( i \in N \setminus (N^0 \cup N^1) \),
\[ \varphi_i(R, M) = \frac{M - n^1 p^1}{n - n^0 - n^1}. \]

**Subcase 1.4.** \( M = n^1 p^1 + (n - n^0 - n^1)p^2 \).

By the same argument as in Subcase 1.3, for each \( i \in N^0 \), \( \varphi_i(R, M) = 0 \), and for each \( i \in N^1 \),
\( \varphi_i(R, M) = p^1 \).

By one-sided composition up, for each \( M' < n^1 p^1 + (n - n^0 - n^1)p^2 \) and each \( i \in N \setminus (N^0 \setminus N^1) \),
\( \varphi_i(R, M) \geq \varphi_i(R, M') = \frac{M' - n^1 p^1}{n - n^0 - n^1}. \]
Since \( \sum_{i \in N \setminus (N^0 \setminus N^1)} \varphi_i(R, M) = M - 0 - n^1 p^1 = (n - n^0 - n^1)p^2 \),
for each \( i \in N \setminus (N^0 \cup N^1) \),
\( \varphi_i(R, M) = p^2 \).
Let \( p^3 \equiv \min_{i \in N \setminus (N_0 \cup N_1 \cup N_2)} p(R_i) \), \( N^3 \equiv \{ i \in N : p(R_i) = p^3 \} \), and \( n^3 \equiv |N^3| \).

**Subcase 1.5.** \( M \in (n^1 p^1 + (n - n^0 - n^1)p^2, n^1 p^1 + n^2 p^2 + (n - n^0 - n^1 - n^2)p^3) \).

\[
\ldots
\]

Repeating the argument, we obtain that if \( M \in [0, \sum_{i \in N} \bar{p}(R_i)] \), then \( \varphi(R, M) = U(R, M) \).

**Case 2:** \( M \in \left( \sum_{i \in N} \bar{p}(R_i), \infty \right) \).

Let \( p^k \equiv \max_{i \in N} p(R_i) \), \( N^k \equiv \{ i \in N, \text{ such that } p(R_i) = p^k \} \), and \( n^k \equiv |N^k| \).

**Subcase 2.1.** \( M \in [np^k, \infty) \).

Suppose that there are \( i, j \in N \) such that \( \varphi_i(R, M) > \frac{M}{n} \) and \( \varphi_j(R, M) < \frac{M}{n} \). Since \( \varphi_i(R, M) > p^k \geq p(R_i) \) and \( \varphi_i(R, M) > \varphi_j(R, M) \), there is \( R'_i \in \mathcal{R} \) such that \( \varphi_j(R, M) P_i' \varphi_i(R, M) \). By peak only, \( \varphi(R'_i, R_{-i}, M) = \varphi(R, M) \). Then, agent \( i \) envies agent \( j \) at \( (R'_i, R_{-i}, M) \). This contradicts no envy. Thus, for each \( i \in N \), \( \varphi_i(R, M) = \frac{M}{n} \).

**Subcase 2.2.** \( M \in \left( \sum_{i \in N} \bar{p}(R_i), np^k \right) \).

The remainder of the proof follows the same logic as that of Theorem 1. □

- **On the independence of the axioms in Theorem 2**

(1) The equal division rule satisfies all the axioms of Theorem 2 except for weak irrelevance of null agents.

(2) The proportional rule satisfies all the axioms of Theorem 2 except for no envy.

(3) Consider the “modified uniform”\(^{11}\) rule defined as follows: for each \( N \subseteq \mathcal{N} \) and each \( (R, M) \in \Pi^N \), let \( p^k \equiv \max_{i \in N} p(R_i) \), \( N^k \equiv \{ i \in N : p(R_i) = p^k \} \), and \( p^{k-1} \equiv \)

\(^{11}\)We call it the “modified uniform” rule because it is the same as the uniform rule except for a few cases.
max_{i \in N \setminus N^k} p(R_i). For each i \in N,

\[ \text{ModU}_i(R, M) = \begin{cases} 
\bar{p}(R_i) + \frac{M - \sum_{j \in N} \bar{p}(R_j)}{|N^k|} & \text{if } \sum_{j \in N} \bar{p}(R_j) < M, \; i \in N^k, \text{ for each } j \in N^k, \; \infty P_j p^{k-1}, \\
\bar{p}(R_i) & \text{if } \sum_{j \in N} \bar{p}(R_j) < M, \; i \notin N^k, \text{ for each } j \in N^k, \; \infty P_j p^{k-1}, \\
U_i(R, M) & \text{otherwise.}
\end{cases} \]

The modified uniform rule satisfies all the axioms of Theorem 2 except for peak only.

(4) The mixed rule satisfies all the axioms of Theorem 2 except for one-sided composition down.

(5) The reverse mixed rule satisfies all the axioms of Theorem 2 except for one-sided composition up.

The uniform rule is the only rule satisfying efficiency, no envy, and peak only (Thomson, 1994a). No combination of any four of the axioms of Theorem 2 implies efficiency. This is shown next.

(1) The equal division rule satisfies no envy, peak only, one-sided composition up, and one-sided composition down, but not efficiency.

(2) Let \( k \in \mathcal{N} \). Let \( \varphi^k \) be defined as follows: for each \( N \subseteq \mathcal{N} \), each \( (R, M) \in \Pi^N \), and each \( i \in N \),

\[ \varphi^k_i = \begin{cases} 
U_i(R, M) & \text{if } \sum_{j \in N} \bar{p}(R_j) \geq M, \\
0 & \text{if } \sum_{j \in N} \bar{p}(R_j) < M, \; \text{and } i = k, \\
U_i(R_{-k}, M) & \text{if } \sum_{j \in N} \bar{p}(R_j) < M, \; \text{and } i \neq k.
\end{cases} \]

The rule \( \varphi^k \) satisfies weak irrelevance of null agents, peak only, one-sided composition up and one-sided composition down, but not efficiency.

(3) Let \( \phi \) be defined as follows: For each \( N \subseteq \mathcal{N} \) and each \( (R, M) \in \Pi^N \), let \( N^0 \equiv \{i \in N : p(R_i) \leq 0\} \), \( p^1 \equiv \min_{i \in N \setminus N^0} p(R_i) \), \( N^1 = \{i \in N : p(R_i) = p^1\} \), and \( p^2 = \min_{i \in N \setminus (N^0 \cup N^1)} p(R_i) \).

The preference profile \( R \) satisfies condition * if there is \( i^* \in N \) such that \( N^0 = \emptyset, N^1 = \{i^*\} \),
0 \ P_i p^2, and for each \( i \in N \setminus \{i^*\} \), \( \propto P_i 0 \). For each \( i \in N \),

\[
\phi_i = \begin{cases} 
U_i(R, M) & \text{if } \sum_{j \in N} \bar{p}(R_j) \geq M, \\
0 & \text{if } \sum_{j \in N} \bar{p}(R_j) < M, \ R \text{ satisfies condition } \ast, \ \text{and } i = i^*, \\
U_i(R_{-i^*}, M) & \text{if } \sum_{j \in N} \bar{p}(R_j) < M, \ R \text{ satisfies condition } \ast, \ \text{and } i \neq i^*.
\end{cases}
\]

The rule \( \phi \) satisfies \textit{weak irrelevance of null agents, no envy, one-sided composition up} and \textit{one-sided composition down}, but not \textit{efficiency}.

(4) The mixed rule satisfies \textit{weak irrelevance of null agents, no envy, and peak only, one-sided composition up}, but not \textit{efficiency}.

(5) The reverse mixed rule satisfies \textit{weak irrelevance of null agents, no envy, and peak only, one-sided composition down}, but not \textit{efficiency}.

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**References**


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