Optimal Equity Auctions with Heterogeneous Bidders (Please Do Not Distribute Externally)

Tingjun Liu*
Cheung Kong Graduate School of Business
2015

Abstract

This paper investigates the design and performance of equity auctions—auctions in which bidders pay with equities instead of cash. Such equity auctions arise in many economic situations including mergers and acquisitions, venture capital financing, and oil and gas lease auctions. Among all incentive-compatible mechanisms of equity auctions, I identify the mechanism that maximizes the seller’s expected revenue. I study how bidder heterogeneity affects the optimal design, and obtain the distinct implications of different sources of the heterogeneity. I show it is optimal for the seller to have smaller bidders win more often, because bidders’ informational advantages decrease in their sizes—and hence the seller can extract a larger proportion of rents from smaller bidders.

Key words: Auctions; Bidder heterogeneity; Equity bids; Mechanism design

JEL classification: D44; D82

*Department of Finance, Cheung Kong Graduate School of Business, Tower E2, 1 East Chang An Avenue, Beijing, China 100738; email: tjliu@ckgsb.edu.cn.
1 Introduction

In equity auctions, rather than pay cash, bidders pay with equities that are claims to a fraction of the future cash flows generated by the combined entity of the bidder and the auctioned asset. Equity auctions are common. Andrade, Mitchell, and Stafford (2001) report 58% of mergers and acquisitions are paid entirely in equity and 70% involve equity. Faccio and Masulis (2005) show that institutional rigidities and liquidity concerns (e.g., tax considerations or cash constraints) can lead to the use of equities. Skrzypacz (2013) reports oil and gas lease auctions typically feature equity payments in the form of royalties. Venture capital financing, procurement auctions, and lead-plaintiff auctions also use securities (whose values depend on realizations of future cash flows) with predominantly equity components.

The equity-auctions literature has focused on ex-ante identical bidders where bidders have the same standalone market values, investment costs, and synergy distributions. In their seminal papers, Hansen (1985) and Riley (1988) obtain an important finding that equity bids can generate higher expected revenues than cash bids; De-Marzo, Kremer, and Skrzypacz (2005) study general securities auctions and derive an elegant irrelevance result that all standard (e.g., first- and second-price) formats of equity auctions yield the same expected revenue when bidders are ex-ante identical.

While the assumption of ex-ante identical bidders simplifies the analysis and provides important insights into the workings of equity auctions, incorporating bidder heterogeneity is of practical importance. Heterogeneity is pervasive in auction settings: bidders usually differ ex ante in their characteristics, such as size or distribution of valuations. In takeover auctions, bidders may differ in their market values. In project-rights auctions, bidders may face different opportunity or financing costs. Moreover, in many situations, bidders’ (e.g., strategic versus financial) valuations come from different distributions (Povel and Singh (2006); Gorbenko and Malenko (2013)).

I show bidder heterogeneity is important for the design and performance of equity auctions–more so than for cash auctions. Key to this is the fact that, unlike cash bids, the monetary values of equity bids are nontransparent: the values not only depend on the bids’ face values (i.e., equity fractions), but also on bidders’ observable characteristics (e.g., market values) and private types (e.g., synergies). For instance, consider a takeover auction in which an acquirer offers a fraction of the merged firm’s equity
upon winning. The offer’s monetary value is this fraction multiplied by the sum of the acquirer’s standalone market value, the target’s standalone market value, and the synergy the acquirer can realize in the target, which is typically the acquirer’s private information. By contrast, the value of a cash offer does not vary with an acquirer’s standalone market value or synergy value.

Reflecting this lack of transparency, bidder heterogeneity impacts the performance of equity auctions. Absent bidder heterogeneity, expected revenue is insensitive to the auction design and standard equity formats always generate higher expected revenues than cash auctions (Hansen (1985); Riley (1988); DeMarzo, Kremer, and Skrzypacz (2005)). In contrast, when bidders differ ex ante, expected revenues for different auction formats can vary widely and depend sensitively on the nature of the heterogeneity. Indeed, standard equity-auction formats can even generate lower revenues than cash auctions. With heterogeneous bidders, the proper design of equity auctions becomes important.

I investigate the optimal design of equity auctions with heterogeneous bidders. To my knowledge, my paper is the first to. I determine the revenue consequences of bidder heterogeneity for different equity-auction forms. Among all incentive-compatible mechanisms of equity auctions, I identify the mechanism that maximizes the seller’s expected revenue. I show how bidder heterogeneity alters the optimal design and derive the distinct implications of different sources of bidder heterogeneity.

In my model, risk-neutral bidders compete to acquire a target in a takeover auction in which payments take the form of equities. Bidders privately observe their synergies with the target which are distributed independently. The bidders’ and target’s standalone market values and the synergy distributions are common knowledge. I solve for the optimal selling mechanism in full generality, incorporating two important sources of heterogeneity: bidders may differ in their market values and synergy distributions. Although I place the model in the context of takeover auctions, the results apply generally to the sale of any indivisible asset through equity payments.

I show optimal equity auctions have five key features. First, their allocations de-
pend on bidders’ market values. Given the same synergy distributions, smaller bidders will win more often. To understand why, note that bidders in auctions earn informational rents for having private information on synergies. If bidders do not retain their full synergies when they win, as in equity auctions, the impact of the synergies on their winning profits reduce, thereby decreasing their informational advantages.\(^3\) When bidders of different sizes participate in an equity auction, smaller bidders will bid greater equity shares and thus retain smaller equity stakes upon winning. Hence, smaller bidders have smaller informational advantages. Thus a seller can extract a larger proportion of their rents, making it optimal to let smaller bidders win more often. This result contrasts sharply with that in optimal cash auctions (Myerson 1981), in which allocations do not depend on bidders’ market values—bidders retains all of their equities so that their informational advantages, and hence the seller’s rents, do not depend on bidder sizes.

For plausible parameterizations, optimal equity and cash auctions can lead to very different allocations. Consider a two-bidder example in which synergies are i.i.d. uniform on \([1,2]\), and the market values of the smaller bidder and the target are both 3. In optimal cash auctions, each bidder is equally likely to win, regardless of the sizes of the bidders. In contrast, when the smaller bidder is half the size of the larger in optimal equity auctions, the winning probability of the smaller bidder is 1.23 times that of the larger, and when the smaller bidder is one quarter the size of the larger, the ratio increases to 1.44. This contrast demonstrates the extent to which allocations in optimal equity auctions favor smaller bidders.

The second key feature of optimal equity auctions is that when bidders’ synergy distributions differ, inefficient allocations result (the highest-synergy bidder sometimes loses),\(^4\) but fewer than in optimal cash auctions. Intuitively, heterogeneity in bidders’ synergy distributions spreads their informational advantages, and inefficient allocations exploit such dispersion. Because equity auctions reduce bidders’ informational advantages—and hence the dispersion—synergy distribution heterogeneity leads to fewer inefficiencies in optimal equity than in optimal cash auctions.

Thus, the two forms of bidder heterogeneity have different impacts on allocative

\(^3\)In cash auctions, the marginal impact is 100%: one part increase in a bidder’s synergy translates into one part increase in its winning profit. By contrast, in equity auctions, the marginal impact is less, which decreases the bidder’s informational advantage. See Myerson (1981), DeMarzo, Kremer, and Skrzypacz (2005), and section 4.2 for more discussions on informational advantages.

\(^4\)Bidders with less dispersed synergies tend to win more often.
efficiency: bidder size heterogeneity makes optimal equity auctions less efficient than optimal cash auctions, whereas the opposite holds for synergy distribution heterogeneity. This reflects that bidder size heterogeneity differentiates bidders’ informational advantages in equity but not in cash auctions, making it optimal to bias the allocation for smaller bidders in equity auctions. By contrast, synergy distribution heterogeneity differentiates bidders’ informational advantages in both types of auctions. Because equity bids reduce bidders’ informational advantages, and hence the differences between them, optimal equity auctions feature fewer inefficiencies.

The third key feature of optimal equity auctions is that losing bidders never pay. If bidders pay upon losing, their payments upon winning would decrease, raising the winner’s retained equity share. Because bidders’ informational advantages scale with the equity stake they retain upon winning, the seller’s expected revenues fall. Thus, having only the winner pay is optimal. This result contrasts with cash auctions in which requiring losing bidders to pay (e.g., all-pay formats) can be optimal. A bidder in a cash auction retains all of its equity upon winning, thereby retaining the full extent of its informational advantage whether or not it would pay upon losing.

Fourth, the seller retains the asset when bidders’ synergies are not sufficiently positive. The intuition resembles that in optimal cash auctions: the rents a seller can extract from a bidder (above the target’s standalone market value) are less than the bidder’s synergy and the seller optimally extracts only positive rents. However, the threshold synergy value is generally lower in optimal equity than in optimal cash auctions. This difference reflects the seller’s ability to extract more rents from equity than cash bids, which lowers the threshold value.

Finally, it is interesting to characterize optimal equity auctions when bidders’ market values are very large or very small. In the limit where bidders’ market values far exceed the target’s market value plus synergies, both the allocation and the seller’s revenue in optimal equity auctions approach those in optimal cash auctions. When bidders’ market values are large, the fraction of equity the winner pays is trivial, so the winner retains almost all of the synergy just as in cash auctions. Differences in bidders’ informational advantages between equity and cash auctions vanish and the optimal equity auction structure approaches that of an optimal cash auction. At the other limit when bidders’ market values are far less than the target’s, optimal equity auctions achieve the first-best outcome—the seller extracts all rents. Bidders offer almost all of their equity if their market value is small, retaining almost none of
their synergies. Bidders’ informational advantages—and hence rents—approach zero, allowing the seller to keep all surplus.

Given these properties, optimal equity auctions respond to both sources of bidder heterogeneity and maximally exploit the features of equity bids. Consequently, they always generate higher expected revenues than optimal cash auctions, regardless of how much bidders differ ex ante. This result formalizes the strong intuition about the advantages of equity bids derived with ex-ante identical bidders. Importantly, the key to the revenue superiority of optimal equity auctions lies in the fact that they *simultaneously* adjust for both forms of bidder heterogeneity; by contrast, any equity-auction format that only adjusts for one form of bidder heterogeneity would generate lower revenues than optimal cash auctions when the other form of bidder heterogeneity is substantial.\(^5\)

In addition to providing insights into the optimal equity auction design, my work also helps in understanding alternative (suboptimal) equity-auction formats. Because optimal equity auctions extract the maximum possible revenue of any equity-auction format, they provide a benchmark for evaluating the merits of alternative formats. My formulation of optimal equity auctions also provides guidance for future research on how standard equity-auction formats can be modified to account for bidder heterogeneity in simple, albeit slightly suboptimal, ways (see section 5). Furthermore, in deriving the optimal design, I obtain a tractable formulation for the expected revenue in any incentive-compatible equity auction. This formulation provides guidance for how the seller should set reserve prices that maximize the performance for any given equity-auction format.

On a technical level, my work contributes to mechanism design methodology. It generalizes the approaches for cash auctions to settings in which payments are equities whose values depend on bidders’ private information. My findings show the associated optimal mechanism is a broader mechanism that subsumes optimal cash auctions as a special limiting case and integrates cash and equity auctions in a unified framework.

My work is also useful for empirical research on bidders’ and seller’s gains in equity auctions. Because optimal equity auctions provide an attainable upper bound on a seller’s proceeds as a function of observable bidder and seller characteristics

---

\(^5\)Existing equity-auction formats do not adjust for synergy distribution heterogeneity. They adjust for size heterogeneity, but the adjustments are suboptimal (even in settings in which bidders differ only in sizes).
(standalone market values and synergy distributions), my results provide a theoretical underpinning to empirical investigations of how observable firm characteristics affect seller-buyer surplus division. In addition, I generalize (Appendix B) the optimal design to settings in which some bidders offer cash and others equity. This generalization facilitates empirical investigations of how offer types also affect the surplus division.

The rest of the paper proceeds as follows. Section 2 reviews the literature. Section 3 describes the model. Section 4 solves for optimal equity auctions. Section 5 derives their properties and implications. Section 6 concludes. Appendix A contains proofs. Appendix B extends the model to incorporate cash payments. Appendix C generalizes the analysis to broader sets of securities. Appendix D provides further details on the analysis.

2 Related Literature


These papers examine the optimal selling mechanism in cash auctions. By contrast, I examine the optimal selling mechanism in equity auctions. What complicates the analysis is that the approaches for deriving optimal cash mechanisms do not directly apply, because of the dependence of equity bids’ values on bidders’ private types. Nonetheless, through a set of transformations on bidders’ incentive conditions and on the seller’s objective function, I show the concept of virtual valuation still holds, and via this concept, the optimal design can again be formulated. The virtual

6Studies have also analyzed revenue consequences in cash auctions in many other situations, for example, preemptive bidding (Fishman (1988)), sequential auctions (Bernhardt and Scoones (1994)), and bidder cross-shareholdings (Dasgupta and Tsui (2004)).
valuation in equity auctions has a finer structure than its cash auction counterpart, and I derive the rich set of implications of this structure.


DeMarzo, Kremer, and Skrzypacz (2005) provide a general analysis of security-bid auctions. They show that when the seller restricts bids to an ordered set and uses a standard auction format, steeper securities yield higher revenues, and the first-price auction with call options yields the highest revenue over a general set of auction mechanisms. They also establish revenue equivalence among standard formats when bidders are ex-ante identical and securities are linear (e.g., equities). Their paper and mine solve for optimal selling mechanisms under orthogonal constraints: they consider ex-ante identical bidders and incorporate general classes of securities, whereas I examine equity auctions and incorporate heterogeneous bidders.

\(^7\)Bidding with cash when bidders have limited liability is similar to bidding with debts.

\(^8\)When bidders have different market values, Hansen (1985) constructs a revised second-price equity auction that adjusts for bidders’ market values and achieves efficient allocations. The expected revenue in this format improves over the standard second-price format. However, as we have shown, when either source of bidder heterogeneity is present, allocational efficiency is not optimal for the seller: optimal equity auctions generate strictly higher revenues than the revised second-price auction.
3 The Model

A group of $n \geq 1$ risk-neutral bidders competes to acquire a target. The target and bidders have standalone market values $V_T$ and $V_i$ ($i = 1, ..., n$). Bidder $i$ values the target at $x_i$, which is $V_T$ plus the synergy gains the bidder can realize upon acquiring the target. Bidder $i$ privately observes $x_i$, independently drawn from cumulative distribution $F_i$ (with p.d.f. $f_i$) with full support on $[x_i, \bar{x}_i]$. The values of $V_T$, $V_i$, and the functional forms of $F_i$ ($i = 1, ..., n$) are common knowledge. The model accommodates heterogeneous bidders: $V_i$ and $F_i$ can be different for each bidder.\(^9\)

**Definition 1** Bidders are ex-ante identical if and only if $V_i = V_j$, $x_i = x_j$, $\bar{x}_i = \bar{x}_j$, and $F_i = F_j$ for all $i, j$. Bidders are heterogeneous if they are not ex-ante identical.

The target is sold via an equity auction, and a Nash equilibrium exists to the auction design.

**Definition 2** In an equity auction, the winner pays with equity of the joint firm, and losers pay with equities of their standalone firms.

If bidder $i$ acquires the target, the value of the joint firm is $V_i + x_i$. The target’s objective is to maximize its expected revenue.

**Discussions.** My focus on equity auctions reflects the practical consideration that equity auctions are widespread and that bidders are often substantially heterogeneous.\(^10\) Because the value of an equity bid depends on both $V_i$ and $x_i$, both sources of bidder heterogeneity (Definition 1) are relevant for equity auctions.

Although I pose the model in the context of takeover auctions, it applies generally to the sale of any non-divisible asset through equity payments. For example, all results hold in project-rights auctions upon replacing the bidder’s market value with its investment cost, and the bidder’s valuation with the sum of the net present value of the project under its control and any investment cost the seller incurs.

In Appendix B, I extend the model and allow bidders to pay with combinations of cash and equity. I show it is optimal for the seller to accept only equity payments with no cash components.\(^11\)

\(^9\)My model also applies to the special case of ex-ante identical bidders, including the case of a single bidder ($n = 1$).

\(^10\)In addition to directly submitting equity bids, equity bidding includes also settings in which bidders submit monetary bids but finance or pay with equities; see Liu (2012).

\(^11\)In another scenario, if the seller could force bidders to submit any securities with no restrictions on their forms, the optimal mechanism would be trivial: the seller could extract full rents.
4 Optimal Mechanisms

Before presenting the technical details of the analysis, I provide examples in section 4.1 to illustrate revenue differences between cash and equity auctions for ex-ante identical and heterogeneous bidders, respectively. I derive the general properties for incentive-compatible mechanisms in section 4.2, and solve for the optimal mechanism in section 4.3.

4.1 Examples with Ex-ante Identical and Heterogeneous Bidders

Example 1 (Ex-ante identical bidders): Two bidders and the target have the same market value $V_1 = V_2 = V_T = 3$. The bidders’ valuations for the target, $x_1$ and $x_2$, are uniformly distributed over $[4, 5]$.

First consider a second-price cash auction: bidders make cash offers, and the highest bidder wins and pays the second-highest offer. Truthful bidding is a dominant strategy: for instance, if $x_1 = 5$ and $x_2 = 4$, bidder 1 bids 5 and wins, paying bidder 2’s bid of 4. Integrating over $x_1$ and $x_2$ yields an expected revenue of 4.33.

Next consider a second-price equity auction: bidders offer fractions of the joint firm, and the highest bidder wins and pays the second-highest fraction. Truthful bidding is again a dominant strategy: the bidder would be indifferent to losing if it wins and pays its own bid. Thus, bidder $i$ will bid $\frac{x_i}{V_i}$ for $i$. For instance, if $x_1 = 5$ and $x_2 = 4$, bidder 1 bids $\frac{5}{8}$ and bidder 2 bids $\frac{4}{7}$. Thus, bidder 1 wins and pays bidder 2’s bid, corresponding to a value of $\frac{4}{7} \times (5 + 3) = 4.57$. Integrating over $x_1$ and $x_2$ yields an expected revenue of 4.53.

This example is based on Hansen (1985) and demonstrates how the second-price equity auction generates higher revenues than the second-price cash auctions with ex-ante identical bidders.$^{12}$ As Hansen (1985) and DeMarzo, Kremer, and Skrzypacz (2005) have shown, the gains come from the fact that with the same equity bid, a higher-type bidder pays more in monetary terms. Thus, equity bids effectively lower the difference between the winner’s valuation and that of the second-highest bidder; because this difference is the seller’s rent, reducing it benefits the seller.

$^{12}$Note that in this example, the second-price cash auction (with no reserve price) is also the optimal cash auction; thus, the second-price equity auction generates higher revenues than cash auctions of any format.
Example 2 (Heterogeneous bidders): The target has a market value of 3, and two bidders have the same market value of 18. One bidder’s valuation for the target is uniformly distributed over $[3, 6]$ and the other’s is uniformly distributed over $[4\frac{1}{4}, 4\frac{3}{4}]$.

The key feature of Example 2 is that bidders differ in their valuation distributions. In cash auctions, Myerson (1981) identifies the optimal mechanism that adjusts for such differences. In this example, these optimal adjustments generate an expected revenue of 4.51. In equity auctions, by contrast, existing theories do not prescribe such adjustments: the best mechanism these theories have for Example 2 is still the standard (first- or second-price) format that does not adjust for the valuation distributions.

Consider the standard second-price equity auction as described in Example 1. Note the seller can maximize the auction’s performance by setting appropriate reserve prices.\textsuperscript{13} However, even with optimal reserve prices (section 10.3 shows how to determine them), the expected revenue is only 4.48, which is lower than that from optimal cash auctions. In other words, the expected revenue from standard second-price equity auctions is always lower than that from optimal cash auctions, regardless of how reserve prices are set.

In a more general setting in which bidders also differ in their sizes, such differences do not affect cash auctions but do further complicate equity auctions, and equity-auction formats that do not properly adjust for bidders’ sizes (in addition to not adjusting for valuation distributions) would generate even worse revenues. Consequently, the need for guidance in the auction design becomes even more urgent.

These examples motivate the following questions: When bidders differ ex ante, what forces drive the auction’s revenues? What is the optimal mechanism that generates the highest expected revenue? How does the optimal mechanism respond to different forms of bidder heterogeneity? Does it always generate higher revenues than optimally designed cash auctions, even if bidder heterogeneity is substantial? I next investigate these questions with a formal analysis.

\textsuperscript{13}A reserve price in equity auctions prohibits a bidder from bidding a fraction below a particular value.
4.2 Properties of Incentive-Compatible Mechanisms

Let \( B \equiv \{1, 2, \ldots, n\} \) denote the set of bidders, let \( f(x) \equiv \prod_{i=1}^{n} f_i(x_i) \) denote the joint density of \( x \equiv (x_1, x_2, \ldots, x_n) \), and for all \( i \in B \), let \( f_{-i}(x_{-i}) \equiv \prod_{k \neq i} f_k(x_k) \) denote the joint density of \( x_{-i} \equiv (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \), and let \( \chi_i \equiv [x_i, \bar{x}_i] \) denote the set of bidder \( i \)'s valuation. Let \( \chi \equiv \times_{k=1}^{n} \chi_k \) denote the product of bidders' valuation sets, and let \( \chi_{-i} \equiv \times_{k \neq i} \chi_k \) denote the product of these sets excluding bidder \( i \). Finally, let \( A \equiv \{0, 1, 2, \ldots, n\} \) denote an augmented set of possible winners, where the element "0" denotes the situation in which no bidder wins (i.e., the target is not sold).

In light of the revelation principle (Myerson (1981)), without loss of generality, consider a direct-revelation mechanism \((W, Q)\) in which bidders report their types, truth-telling is an equilibrium, and (1) \( W : x \mapsto \mathbb{R}^{n+1}_{\geq 0} \) is the winning rule, where \( \sum_{j=1}^{n+1} W_j(x) = 1 \) for all \( x \), and for all \( j \in A \), \( W_j(x) \) is bidder \( j \)'s winning probability when bidders report valuations \( x \) (recall that \( j = 0 \) denotes the situation in which no bidder wins, and thus \( W_0(x) \) is the probability that the asset is not sold); and (2) \( Q : \chi \times B \times A \to \mathbb{R}^{n}_{\geq 0} \) is the equity-retention rule: for all \( i \in B \) and \( j \in A \), \( Q_i(x, j) \) is the fraction of equity bidder \( i \) retains (i.e., \( 1 - Q_i \) is the equity fraction bidder \( i \) pays) when bidders report \( x \) and bidder \( j \) wins, and

\[
Q_i(x, j) \in [0, 1]
\]

for all \( x \).

Note the equity-retention rule depends explicitly on the winner's identity. This dependence allows for mechanisms that probabilistically determine the winner, and for mechanisms in which losing bidders pay.

Denote by \( v_i(x_i, z_i) \) bidder \( i \)'s expected profit when it has \( x_i \) but reports \( z_i \), and all other bidders report truthfully; then

\[
v_i(x_i, z_i) = \int_{\chi_{-i}} [(V_i + x_i) Q_i(z_i, x_{-i}, i) - V_i] W_i(z_i, x_{-i}) f_{-i}(x_{-i}) dx_{-i} + \sum_{j \neq i} \int_{\chi_{-i}} [V_i Q_i(z_i, x_{-i}, j) - V_i] W_j(z_i, x_{-i}) f_{-i}(x_{-i}) dx_{-i}.
\]

(2)

To express equation (2) more concisely, denote by \( G_i(z_i) \) bidder \( i \)'s winning proba-
bility when it reports \( z_i \) and all others report truthfully:

\[
G_i (z_i) = \int_{x_{-i}} W_i (z_i, x_{-i}) f_{-i} (x_{-i}) \, dx_{-i}. \tag{3}
\]

Analogously, denote by \( q_i (z_i) \) the expected fraction of the merged entity that bidder \( i \) retains contingent upon winning if it reports a value \( z_i \) and all others report truthfully:

\[
q_i (z_i) G_i (z_i) = \int_{x_{-i}} Q_i (z_i, x_{-i}, i) W_i (z_i, x_{-i}) f_{-i} (x_{-i}) \, dx_{-i}. \tag{4}
\]

Using equations (3) and (4), bidder \( i \)'s expected profit, equation (2), becomes

\[
v_i (x_i, z_i) = [(V_i + x_i) q_i (z_i) - V_i] G_i (z_i) - \omega_i (z_i), \tag{5}
\]

where

\[
\omega_i (z_i) \equiv \sum_{j \neq i} \int_{x_{-i}} V_i (1 - Q_i (z_i, x_{-i}, j)) W_j (z_i, x_{-i}) f_{-i} (x_{-i}) \, dx_{-i}. \tag{6}
\]

The first term on the right-hand side of equation (5) is bidder \( i \)'s expected profit without considering payments upon losing, and the second term, \( \omega_i (x_i) \), is the reduction in the expected profit from payments upon losing.

Denoting bidder \( i \)'s equilibrium expected profit by

\[
u_i (x_i) \equiv v_i (x_i, x_i), \tag{7}\]

the incentive compatibility condition gives

\[
u_i (x_i) = \max_{z_i} v_i (x_i, z_i), \tag{8}\]

which, by (5) and the general envelop theorem (see Milgrom and Segal (2002)), yields

\[
u_i (x_i) = u_i (x_i) + \int_{z_i}^{x_i} q_i (t) G_i (t) \, dt. \tag{9}\]

Equation (9) is instructive. Note the term \( q_i \) is less than 1 in equity auctions, whereas it equals 1 in cash auctions. Thus, (9) shows that the differential rents a high-valuation bidder earns over a low-valuation bidder in equity auctions is less than
in cash auctions, which, under appropriate boundary conditions, reduces bidders’ overall rents and raises the seller’s revenue. This reflects the following insights that Hansen (1985) and DeMarzo, Kremer, and Skrzypacz (2005) provide: because equity (or general security) bids tie payments to the winner’s actual value, a higher-type bidder will pay more even with the same bid; hence, the linkage principle of Milgrom and Weber (1982) implies the seller benefits.

Equation (9) also shows that a bidder’s (differential) rent in fact is proportional to \( q_i \), which represents the extent to which the bidder retains its valuation upon winning.\(^ {14} \) This scaling property is helpful for understanding many of the results of this paper.

Next, let

\[
\pi_{s,i} = \int_{x_i}^{x_i} G_i(x_i) (x_i - V_T) f_i(x_i) \, dx_i - \int_{x_i}^{x_i} u_i(x_i) f_i(x_i) \, dx_i. \tag{10}
\]

The first term is bidder \( i \)’s contribution to the expected increase in social welfare, the second term is the bidder’s expected profit, and their difference, \( \pi_{s,i} \), corresponds to the bidder’s contribution to the seller’s expected profit. Summing over these contributions yields the seller’s total expected profit, which, upon adding \( V_T \), gives the seller’s expected revenue:

\[
\pi_s = \sum_{i=1}^{n} \pi_{s,i} + V_T. \tag{11}
\]

Plugging in equation (9), the second term in equation (10) becomes

\[
\int_{x_i}^{x_i} u_i(x_i) f_i(x_i) \, dx_i = \left[ u_i(x_i) + \int_{x_i}^{x_i} q_i(t) G_i(t) \, dt \right] \left[ (1 - F_i(x_i)) \right] = u_i(x_i) + \int_{x_i}^{x_i} (1 - F_i(x_i)) q_i(x_i) G_i(x_i) \, dx_i.
\]

\(^ {14} \)Intuitively, a high-valuation bidder can always deviate by mimicking a low-valuation bidder and, upon winning, enjoy higher rents (due to its higher valuation) than the low-valuation bidder. Such a rent difference scales with the extent to which the bidder would retain its valuation upon winning. To prevent such a deviation, in equilibrium, the differential rents a high-valuation bidder earns over a low-valuation bidder scale accordingly.
where integration by parts is used. Thus, equation (10) becomes

$$
\pi_{s,i} = \int_{x_i}^{x_i} G_i (x_i) (x_i - V_T) f_i (x_i) dx_i - \int_{x_i}^{x_i} (1 - F_i (x_i)) q_i (x_i) G_i (x_i) dx_i - u_i (x_i).
$$

(12)

Note the right-hand side of (12) depends on $q_i (x_i)$. In cash auctions, $q_i (x_i)$ is always 1, independent of $x_i$. However, when bidders pay with equities as in this study, $q_i (x_i)$ is less than 1 and is not constant. This feature, although it complicates the analysis, will lead to a rich set of new implications for equity auctions.

Plugging (5) into the right-hand side of (7) and equating with the right-hand side of (9), one has

$$
[(V_i + x_i) q_i (x_i) - V_i] G_i (x_i) - \omega_i (x_i) = u_i (x_i) + \int_{x_i}^{x_i} q_i (t) G_i (t) dt,
$$

(13)

where $\omega_i (\cdot)$ is given by (6). Equation (13) imposes a constraint on the function $q_i$ in the form of an integral equation. Note $q_i$ is a weighted average of the primitive functions of the mechanism, $W_i$ and $Q_i$, via equation (4); thus, in effect, equation (13) places a constraint on $W_i$ and $Q_i$.

Through a set of transformations on the constraint (13) and on the seller’s objective function, the following theorem eliminates the function $q_i$ from (12) and expresses the optimization program in a form that allows for the explicit solution of the optimal design.

**Theorem 1 (Revenue decomposition and the existence of virtual valuation in equity auctions):** In any incentive-compatible mechanism of equity auction, the seller’s expected revenue (equation 11) decomposes:

$$
\pi_s = \pi_{s,a} + \pi_{s,b} + \pi_{s,c},
$$

(14)

where

$$
\pi_{s,a} = - \sum_{i=1}^{n} u_i (x_i) \left\{ \frac{1}{(V_i + x_i)} \int_{x_i}^{x_i} (1 - F_i (x_i)) dx_i + 1 \right\},
$$

(15)

$$
\pi_{s,b} = - \sum_{i=1}^{n} \int_{x_i}^{x_i} (1 - F_i (x_i)) \left\{ \frac{\omega_i (x_i)}{V_i + x_i} + \int_{x_i}^{x_i} \frac{\omega_i (t)}{(V_i + t)^2} dt \right\} dx_i,
$$

(16)
\[ \pi_{s,c} \equiv \int_{x} \left[ \sum_{i=1}^{n} W_i(x) \phi_i(x_i) + W_0(x) V_T \right] f(x) \, dx, \tag{17} \]

and \( \phi_i(x_i) \) is the virtual valuation defined in equation (18).

**Definition 3** The virtual valuation in equity auctions is

\[
\phi_i(x_i) \equiv x_i - \frac{V_i(1 - F_i(x_i))}{(V_i + x_i) f_i(x_i)} - \frac{V_i \int_{x_i}^{x} (1 - F_i(t)) \, dt}{(V_i + x_i)^2 f_i(x_i)}. \tag{18} 
\]

The three terms in Theorem 1 represent contributions to the expected revenue from bidders’ rationality constraints, losing bidders’ payments, and the auction’s allocations, respectively. The theorem shows these contributions are separable, similar to the revenue decomposition in cash-auction settings (Myerson (1981)). Because \( u_i(x_i) \geq 0 \) (required by the bidder’s individual rationality constraint) and \( \int_{x_i}^{x} (1 - F_i(x_i)) \, dx_i + 1 > 0 \), the maximum possible value for the first term \( \pi_{s,a} \) is zero, which obtains if \( u_i(x_i) = 0 \) for all \( i \). Also because all terms on the right-hand side of (6) are nonnegative, \( \omega_i(x_i) \geq 0 \) and \( \frac{\omega_i(x_i)}{V_i + x_i} + \int_{x_i}^{x} \frac{\omega_i(t)}{(V_i + t)^2} \, dt \geq 0 \) for all \( x_i \). Thus the maximum possible value for the second term \( \pi_{s,b} \) is zero, which obtains if losing bidders do not pay; that is, \( Q_i(x, j) = 1 \) for all \( i, x \), and \( j \neq i \). Next, consider the term \( \sum_{i=1}^{n} W_i(x) \phi_i(x_i) + W_0(x) V_T \), which equals \( 1 + \sum_{i=1}^{n} W_i(x) (\phi_i(x_i) - V_T) \), in \( \pi_{s,c} \).

The allocation \( W_i(X) \) is, in effect, a weighting function, and hence giving weight only to the maximal \( \phi_i(x_i) \) is optimal, provided that \( \phi_i(x_i) \) exceeds the target’s reservation value \( V_T \). This would maximize the expression at every point \( x \) and thus would maximize the third term \( \pi_{s,c} \).

To better understand Theorem 1, consider the corresponding cash-auctions result for comparison. In deriving optimal cash auctions, Myerson (1981) develops the concept of virtual valuation, showing that it represents the rents the seller can extract from a bidder. Myerson (1981) obtains the form of the virtual valuation as a function of the bidder’s actual valuation and its distribution:

\[
\psi_i(x_i) \equiv x_i - \frac{1 - F_i(x_i)}{f_i(x_i)}. \tag{19} 
\]

When bidders offer securities, on the other hand, because of the complication that security bids’ values depend on bidders’ private types, the classic mechanism-design approaches do not directly apply, and a priori, whether virtual valuations can even
be defined in such settings is unclear.

Theorem 1 demonstrates an important result that the concept of virtual valuation holds in equity auctions; namely, a function exists for an equity-offering bidder in terms of the bidder’s private type and any common knowledge (e.g., bidders’ market values and valuation distributions), which measures the rents the seller can extract from the bidder. Equation (18) specifies the form of the virtual valuation, and section 4.3 shows optimal equity auctions can again be formulated via this solution concept. The virtual valuation for equity auctions entails a finer structure than its cash-auction counterpart, with its additional dependence on the bidder’s market value. In section 5, I show this structure leads to a rich set of implications and is crucial for understanding how bidder heterogeneity alters the optimal auction design. In Appendix C, I show virtual valuations can also be defined in a broader class of securities than equities.

Note that Theorem 1 relates to the elegant result of DeMarzo, Kremer, and Skrzypacz (2005) that all symmetric and increasing mechanisms of equity auctions generate the same expected revenues. Intuitively, the existence of virtual valuations in equity auctions, as Theorem 1 establishes, implies that the expected revenue in any equity auction depends only on the allocations (under certain boundary conditions), which is consistent with the insights of DeMarzo, Kremer, and Skrzypacz (2005).

4.3 Solutions for Optimal Mechanisms

I now derive optimal mechanisms when the design problem is regular.

Assumption 1 The design problem is regular: \( \phi_i(\cdot) \) strictly increases over \([x_i, \bar{x}_i]\) for all \(i\).

The regularity condition generally holds as long as the distribution \(f_i(\cdot)\) does not decrease too quickly. This condition is typically easier to satisfy than its analogue in cash auctions for the following reason. In both equity and cash auctions, a bidder’s virtual valuation is smaller than its actual valuation \(x_i\), and the difference represents the bidder’s rents. Because the valuation increases in itself at a rate of one, the regularity condition holds unless this difference (between the valuation and the virtual

15 In addition to providing insights into the optimal auction design, Theorem 1 also provides guidance for how to set optimal reserve prices for alternative (suboptimal) formats to maximize their performance (see Corollary 2 in section 4.3 or Lemma 6 in Appendix D, for instance).
valuation) decreases in \( x_i \) with a rate of more than one. Because the bidder captures less rent in equity auctions, this difference—and its rate of change—is generally smaller, and hence the regularity condition is more likely to hold. In particular, when the bidder is much smaller than the seller, the seller extracts almost full rents from the bidder (Corollary 4) and the regularity condition holds for any distribution \( F_i(\cdot) \).

I assume the regularity condition holds in the rest of the paper (Assumption 2 of section 5 provides a sufficient condition for the regularity condition), which simplifies the analysis. Under this condition, an individually rational and incentive-compatible mechanism exists that maximizes \( \pi_{s,a} \), \( \pi_{s,b} \), and \( \pi_{s,c} \) simultaneously; it follows that the mechanism maximizes the sum of \( \pi_{s,a} \), \( \pi_{s,b} \), and \( \pi_{s,c} \) and is therefore optimal.

I first present a necessary condition for the optimal mechanism.

**Lemma 1** Losing bidders never pay in the optimal mechanism: \( Q_i(x, j) = 1 \) for all \( i \neq j \).

The intuition for Lemma 1 reflects the fact that such payments typically increase the amount of equity a bidder retains upon winning. As informational rents scale with the equity retention, the bidder’s rents increase, thereby reducing the seller’s revenues. This result contrasts with optimal cash auctions, in which losing bidders may also pay (e.g., all-pay auctions),\(^{16}\) because bidders in cash auctions retain all of their equities upon winning; hence, whether they would pay upon losing does not affect their informational advantages.

I now formulate optimal equity auctions.

**Proposition 1** (The set of all optimal mechanisms): A direct-revelation mechanism \((W, Q)\) is optimal if and only if

(i) the winning rule is

\[
W_i(x) = \begin{cases} 
1 & \text{if } \phi_i(x_i) > \max_{j \neq i} \{\phi_j(x_j)\} \text{ and } \phi_i(x_i) \geq V_T \\
0 & \text{if } \phi_i(x_i) < \max_{j \neq i} \{\phi_j(x_j)\} \text{ or } \phi_i(x_i) < V_T 
\end{cases}
\]

where \( \phi_i(x_i) \) is the virtual valuation in equation (18), for all \( i \) and \( x \),

\(^{16}\)To see how optimal cash auctions allow losing bidders to pay, note a bidder’s payment function in optimal cash auctions is determined only up to its expected value, which sums over its winning and losing payments. Thus, optimal cash auctions accommodate losing bidders’ payments, as long as the winning payments are adjusted accordingly. For details on optimal cash auctions, see Myerson (1981) and Krishna (2010).
(ii) losing bidders do not pay, and
(iii) the equity-retention rule upon winning satisfies

\[
\int_{x_{-i}} Q_i(x_i, x_{-i}, i) W_i(x_i, x_{-i}) f_{-i}(x_{-i}) \, dx_{-i} = \frac{V_i G_i(x_i)}{V_i + x_i} + \int_{x_i}^{x_i} \frac{V_i G_i(t)}{(V_i + t)^2} \, dt \tag{21}
\]

for all \(i\) and \(x_i\), where \(G_i(\cdot)\) is given by (3), and (1) holds for all \(i, x, j\).

**Corollary 1** (An optimal mechanism) The following constitutes an optimal mechanism: the winning and payment rules in (i) and (ii) of Proposition 1, and the following equity-retention rule upon winning

\[
Q_i(x, i) = \frac{V_i}{V_i + y_i(x_{-i})}, \tag{22}
\]

where

\[
y_i(x_{-i}) \equiv \phi_i^{-1}\left(\max \left\{ \max_{j \neq i} \{ \phi_j(x_j) \}, V_T \right\} \right) \tag{23}
\]

is the minimum value of \(x_i\) that corresponds to a virtual valuation exceeding \(V_T\) and allows \(i\) to win against \(x_{-i}\), and \(\phi_i^{-1}(\cdot)\) denotes a bounded inverse of \(\phi_i(\cdot)\):

\[
\phi_i^{-1}(x) \equiv \begin{cases} 
  x_i & \text{if } x < \phi_i(x_i) \\
  \bar{x}_i & \text{if } x > \phi_i(x_i) \\
  y \in [x_i, \bar{x}_i] & \text{s.t. } \phi_i(y) = x & \text{if } x \in [\phi_i(x_i), \phi_i(\bar{x}_i)]
\end{cases} \tag{24}
\]

Part (i) of Proposition 1 shows optimal mechanisms select the bidder with the highest virtual valuation as the winner, provided the highest virtual valuation exceeds the target’s reservation value \(V_T\). This result reflects the intuition that the virtual valuation represents the rents the seller can extract from a bidder (equation (17)), and it is optimal to extract only rents exceeding the seller’s reservation value. Section 5 discusses virtual valuation and the properties of optimal mechanisms in detail.

Part (ii) of Proposition 1 shows losing bidders never pay in optimal mechanisms, consistent with Lemma 1. Part (iii) of Proposition 1 (equation (21)) establishes a constraint on the equity-retention rule upon winning, which ensures the incentive compatibility of the mechanism and the boundary condition required by optimality (i.e., \(u_i(\bar{x}_i) = 0\)). Note that equation (21) does not restrict to a specific form the equity-retention rule upon winning. Thus, optimal mechanisms can be constructed in multiple ways with different forms for the equity-retention rule upon winning, as
long as equation (21) holds. Equation (22) in Corollary 1 is one such construction; the mechanism is analogous to the standard second-price auction in that the winner’s payment depends on the valuation of the bidder with the second-highest virtual valuation, and truth-telling is a weakly dominant strategy.

**Implementations.** The optimal mechanisms in Proposition 1 are derived using the revelation principle, and hence they take a form in which bidders report their values. It would be interesting to explore how to implement the optimal mechanisms in auctions in which bidders bid equity fractions they would pay, and the auction rules retain some of the key features of standard auctions.

In the special case in which bidders are ex-ante identical, implementations of optimal equity auctions are particularly simple: all standard equity auctions (in which losers do not pay) with an optimal reserve price are optimal.

**Corollary 2**

(i) When bidders are ex-ante identical, standard first- and second-price equity auctions, in which bidder $i$ must bid a fraction of at least $\phi_i^{-1}(V_T) \frac{V_T}{V_i + \phi_i^{-1}(V_T)}$, are optimal.

(ii) If there is only a single bidder ($n = 1$), making a take-it-or-leave-it offer at $\phi_i^{-1}(V_T) \frac{V_T}{V_i + \phi_i^{-1}(V_T)}$ is optimal.

When bidders are heterogeneous, implementations of optimal equity auctions are more complicated. Nonetheless, they can still be implemented in ways analogous to standard first- or second-price auctions (which I investigate in an extension).

## 5 Properties and Implications

The virtual valuation $\phi_i(x_i)$ in equation (18) is a central component of this paper’s analysis. Below, I first examine its limiting behaviors, and then I investigate its general properties and derive their implications on the optimal design.

**Corollary 3** $\lim_{x_i \to \infty} \phi_i(x_i) = \psi_i(x_i)$, where $\psi_i(x_i)$ is the virtual valuation for cash auctions.

Corollary 3 reflects the fact that the fraction of equity a large bidder would pay upon winning is close to zero; thus, the bidder would retain almost the full extent of its valuation, thereby earning the same rents as in cash auctions. Corollary 3 adds flexibility to the results on the optimal design. In optimal equity auctions, any winning bidder pays with equity. However, Corollary 3 suggests the limiting situation of
$V_i \to \infty$ corresponds to bidder $i$ paying with cash. Thus, upon taking such limits, the formulation of optimal equity auctions applies to settings in which some bidders must offer cash (see section 8.2 for a formal treatment). In particular, if all bidders’ market values are large, optimal equity auctions degenerate to optimal cash auctions: both the allocation and the seller’s revenue in optimal equity auctions approach those in optimal cash auctions.

Corollary 3 corresponds to the lower bound on the expected revenues that optimal equity auctions can generate. In another limiting scenario, the upper bound obtains:

**Corollary 4** \( \lim_{V_i \to 0} \phi_i (x_i) = x_i \).

Corollary 4 represents the first-best outcome of full rent extraction. Intuitively, a small bidder would offer nearly 100% of equity and thus retain almost none of its private valuation, allowing the seller to extract close to full surplus. This full-extraction result is related to Cremer’s (1987) result of full extraction with sufficiently negative cash transfers, because negative cash transfers effectively reduce bidders’ market values.

In the general case in which bidders’ market values are finite, \( \phi_i (x_i) \) is given by equation (18). To better understand (18), note that in both cash and equity auctions, the virtual valuation is less than \( x_i \) and the difference represents the bidder’s rents. In cash auctions, this difference is \( \frac{1 - F_i (x_i)}{f_i (x_i)} \). In equity auctions, this difference has two components, which are the second and third terms in (18). I examine these two terms below.

The second term in the virtual valuation for equity auctions is the second term in the virtual valuation for cash auctions scaled by a factor \( \frac{V_i}{V_i + x_i} \). This factor is the fraction of equity the bidder would keep in order to break even upon winning. It represents a "royalty-rate" effect: consider a setting of constant royalty rates in which the winner pays with a combination of equity and cash, where the equity fraction is \( \alpha_i \) that is fixed for each bidder. Similar algebra as in section 4.2 shows the virtual valuation exists in this setting and is \( \phi_i (x_i) = x_i - (1 - \alpha_i) \frac{1 - F_i (x_i)}{f_i (x_i)} \), where \( (1 - \alpha_i) \) is the equity fraction the winner keeps. Intuitively, if a bidder retains a fixed fraction of equity upon winning, its informational advantage—and hence its rents—scale proportionally.\(^{17}\)

\[^{17}\text{The form of the virtual valuation with constant royalty rates can also be understood by the following limiting result of (18): it approaches } x_i - \frac{V_i}{V_i + x_i} \frac{1 - F_i (x_i)}{f_i (x_i)} \text{ when } V_T \text{ and } V_i \text{ are much larger than the synergy } x_i - V_T \text{. Intuitively, under such a limit, the fraction of equity the bidder retains is a constant } \frac{V_i}{V_i + V_T}, \text{ independent of the bidder’s valuation.}\]
The factor $\frac{V_i}{V_i + x_i}$ endows the virtual valuation with a scaling property, making the virtual valuation decrease in the extent to which the bidder would retain its valuation upon winning. This scaling property is the driving force for the limiting behaviors of the virtual valuation examined earlier, and it has additional implications I will describe.

The third term in the virtual valuation is a correction to the fact that the actual fraction of equity the bidder keeps upon winning is less than $\frac{V_i}{V_i + x_i}$ and is not a constant. This term is particularly sensitive to the upper tails in the valuation distribution: from (18), the ratio of this term to the second term in the virtual valuation is $\frac{1}{V_i + x_i}$ multiplied by a factor, $\frac{1}{1 - F_i(x)} \int_{x_i}^{\bar{x}_i} (1 - F_i(t)) \, dt$, where the factor measures the extent of information asymmetry, or the "fatness" of the tails, in the upper portion of the valuation distribution.

This third term is a unique feature of equity auctions that differs from cash auctions. Its sensitivity to the high tails reflects the underlying difference between equity and cash auctions in that equity bids tie payments to the winner’s actual value, whereas cash bids do not. Thus, the seller’s rents, and hence the virtual valuations, are more sensitive to the distribution tails in equity than in cash auctions. Section 10.2 provides further intuition for such tail effects using a monopolist-pricing analogy.

In section 5.1, I show a sufficiently fat upper tail can cause the third term in the virtual valuation for equity auctions to dominate the second term and impact the optimal auction design. For typical valuation distributions, however, the extent of fat tails tends to be small, and this third term represents only a small correction. Below is a simple condition on the valuation distribution that restricts the size of the fat tails.

**Assumption 2** The distribution function $f_i(\cdot)$ is nondecreasing over $[x_i, \bar{x}_i]$ for all $i$.

This condition also guarantees the regularity condition in Assumption 1:

**Lemma 2** The design problem is regular under Assumption 2.

Below, I derive a number of results under Assumption 2 (in the working paper version, I show these results also hold under more general conditions).

**Lemma 3** Under Assumption 2, for any valuation $x_i$ such that $\psi_i(x_i) \geq V_T$, bidder $i$’s virtual valuation for equity auctions exceeds its cash-auction counterpart, i.e., $\phi_i(x_i) > \psi_i(x_i)$. 

21
Lemma 3 reflects the intuition that equity bids allow the seller to extract more rents than cash bids, which increases the virtual valuation. This result has implications for trading probabilities in optimal auctions. In both optimal equity and optimal cash auctions, if the highest virtual valuation among all bidders is below the seller’s reservation value $V_T$, the seller retains the asset even when bidders’ actual valuations exceed $V_T$ and a trade would result in social gains. Because virtual valuations are higher in equity auctions than in cash auctions as Lemma 3 shows, socially beneficial trade occurs more frequently in optimal equity auctions:

**Proposition 2** Under Assumption 2, the probability that the asset is sold in optimal equity auctions is at least as high as in optimal cash auctions.

The virtual valuation for equity auctions has another feature that differs from its cash-auction counterpart: it depends on the bidder’s market value. From equation (18), $\phi_i(x_i)$ depends on $V_i$ in both the second and third terms. The second term (with its minus sign) decreases in $V_i$ for all values of $x_i$; the third term, however, may either decrease or increase in $V_i$, depending on the value of $x_i$. When the third term is small, $\phi_i(x_i)$ decreases in $V_i$ for all values of $x_i$.

**Lemma 4** Under Assumption 2, for any bidder $i$ and $x_i$ such that $\psi_i(x_i) \geq V_T$, \[
\frac{\partial \phi_i(x_i)}{\partial V_i} < 0.
\]

**Proposition 3** Suppose there are $n \geq 2$ bidders and two of them have the same valuation and valuation distribution, but their market values differ. Under Assumption 2, the smaller bidder has a higher winning probability in optimal equity auctions than the larger one.

The intuition for Lemma 4 and Proposition 3 reflect that upon winning, a smaller bidder pays a larger equity fraction and hence retains a smaller equity stake. Because bidders’ informational advantages scale with the equity retention, a seller can extract more rents from smaller bidders, which makes it optimal to let them win more often.

Proposition 3 sharply contrasts with the results in optimal cash auctions, in which the allocations do not depend on bidders’ market values, because all bidders in cash auctions retain 100% equity stakes—hence the full extent of their informational advantages, independently of their sizes. For plausible parameterizations, optimal equity and optimal cash auctions can lead to significantly different allocations. For instance,
consider a two-bidder case in which valuations are i.i.d. uniform on [4,5] and the market values of the smaller bidder and the target are both 3. Each bidder is always equally likely to win in optimal cash auctions, but when the smaller bidder is half the size of the larger bidder in optimal equity auctions, the winning-probability ratio of the smaller bidder over the larger is 1.23, and when the smaller bidder is one quarter the size of the larger, the ratio increases to 1.44. Appendix D provides more details on the comparisons. These results highlight the extent to which allocations of optimal equity auctions can favor smaller bidders (relative to optimal cash auctions).\(^{18}\)

In addition to responding to bidders’ market values, optimal equity auctions also respond to bidders’ valuation distributions. Suppose bidders have the same market values and their valuations for the target have the same expected value but different dispersion, those bidders with less dispersed valuations are more likely to win (Appendix D provides numerical illustrations). More generally, similar to what Myerson (1981) demonstrates in cash-auction settings, stochastic dominance in the upper tail of valuation distribution is relevant: the allocation tends to favor bidders whose distributions are conditionally first-order stochastically dominated. Intuitively, the seller’s knowledge of such bidders’ valuations is more precise, effectively giving the seller an outside option that allows the seller to demand higher bids from other bidders. This intuition is much the same as that for setting a high reserve price, which allows a seller to extract more rents despite an efficiency loss.\(^{19}\)

The response of optimal equity auctions to bidders’ differing valuation distributions is in the same general direction as that of optimal cash auctions, because virtual valuations in both auctions share the same term, \(\frac{1-F_i(x)}{f_i(x)}\). However, the magnitudes differ: because of the scaling factor \(\frac{V_i}{V_i+x_i}\), optimal equity auctions typically result in fewer inefficiencies than optimal cash auctions. Intuitively, in equity auctions, the seller can extract more rents from bidders; hence, virtual valuations tend to be closer to bidders’ actual valuations, and differences in them due to bidders’ differing valuation distributions fall. As the seller optimally selects the winner based on the virtual valuations, optimal equity auctions generally lead to more efficient allocations than

\(^{18}\)In an extension I show how this property of optimal equity auctions provides insights into the performance of an alternative format of equity auction, which naturally lets smaller bidders win more often, achieving similar allocations and almost as much revenues as optimal equity auctions.

\(^{19}\)An alternative way to understand the intuition is to note that bidders whose valuations are more precisely known have fewer informational advantages; hence, letting these bidders win more often allows the seller to extract a larger proportion of their rents.
optimal cash auctions.

A natural way to rank the efficiencies of two mechanisms is to compare the corresponding social welfare. Below, I define a stronger notion of efficiency ordering: one mechanism is *ex-post more efficient* than another if it leads to higher social welfare at all valuation realizations.

**Definition 4** Let $A$ and $B$ be two mechanisms in which ties occur with zero probability. Given bidders’ valuation distributions, mechanism $A$ is *ex-post more efficient* than $B$ if the following holds at all valuation realizations $(x_1, x_2, ..., x_n)$: (i) if the asset is sold in $B$, it is also sold in $A$, and the winner’s valuation is no less than the valuation of the winner in $B$; (ii) if the asset is not sold in $B$, either the asset is not sold in $A$ or the winner in $A$ has a valuation exceeding the target’s reservation value $V_T$.

I show optimal equity auctions are ex post more efficient than optimal cash auctions under the following conditions (Appendix D provides numerical illustrations):

**Proposition 4** Suppose bidders have the same market values. Optimal equity auctions are ex-post more efficient than optimal cash auctions if (i) bidders’ valuations are uniformly distributed (their support may differ), or (ii) bidders’ and target’s market values are much larger than the extent of bidders’ synergies; that is, given any synergy distributions and constants $V^*_T$ and $V^*$, let $V_T = kV^*_T$ and $V_i = kV^*$, $i = 1, ..., n$, where $k$ is arbitrarily large, or (iii) bidders’ market values are much smaller than the target’s.

Note the two forms of bidder heterogeneity have opposite impacts on the efficiency of the optimal auction: heterogeneity in bidder sizes makes optimal equity auctions less efficient than optimal cash auctions, whereas heterogeneity in distributions of bidder valuations leads to more efficient allocations. To understand this contrast, observe that bidder heterogeneity spreads bidders’ informational advantages—and inefficient allocations exploit such dispersion. As heterogeneity in valuation distributions affect bidders’ informational advantages in both cash and equity auctions, and equity auctions reduce bidders’ informational advantages—and hence the degree of the dispersion, optimal equity auctions lead to more efficient allocations than optimal cash auctions when bidders differ only in valuation distributions. By contrast, a bidder’s market value affects its informational advantage in equity but not in cash.
auctions, making it attractive to favor smaller bidders (with less informational advantages) in equity auctions. Consequently, optimal equity auctions result in less efficient allocations than optimal cash auctions when bidders differ only in market values.

Optimal equity auctions account for both sources of bidder heterogeneity and maximally exploit the features of equity bids. Consequently, they generate higher expected revenues than cash auctions of any format.

**Proposition 5** *Optimal equity auctions generate strictly higher expected revenues than optimal cash auctions, regardless of any heterogeneity in bidders.*

In symmetric settings with ex-ante identical bidders, the revenue advantages of equity over cash auctions have been well established in the literature (Hansen (1985); Riley (1988); DeMarzo, Kremer, and Skrzypacz (2005); among others). Proposition 5 generalizes the revenue advantages of equity auctions to settings with heterogeneous bidders. Importantly, with heterogeneous bidders, the key to the revenue superiority of optimal equity auctions lies in the fact that they simultaneously adjust for both sources of bidder heterogeneity. By contrast, any equity-auction format that adjusts for only a single source of bidder heterogeneity can generate revenues lower than optimal cash auctions when the other source of bidder heterogeneity is substantial.

### 5.1 Abnormalities

Lemmas 3 and 4 and Propositions 2 and 3 are established under Assumption 2, which is a sufficient condition that restricts the “fatness” of the upper tails of valuations. One might conjecture these results follow from the casual intuition that the seller can always extract more rents (i) from equity bids than cash bids and (ii) from smaller bidders than larger bidders in equity auctions, and hence that no conditions need be imposed to ensure these results. I show such conjectures are false.

Concretely, I show the following can obtain even when $\phi_i(\cdot)$ and $\psi_i(\cdot)$ monotonically increase on $[\bar{x}_i, \bar{x}_i]$:

1. A virtual valuation in the cash auction can exceed that in the equity auction; that is, $\psi_i(x^*) > \phi_i(x^*) \geq 0$ for some $x^*$.

2. The probability of trade can be lower in the optimal equity auction than in the optimal cash auction.
3. A bidder’s virtual valuation can increase in its size; that is, \( \frac{\partial \phi_i(x^*)}{\partial V_i} > 0 \) and \( \phi_i(x^*) > 0 \) for some \( x^* \). That is, valuation and valuation distribution exist, under which larger bidders in optimal equity auctions are more likely to win.

Appendix D provides details on these seemingly counter-intuitive results, showing how they arise when the valuation distribution has sufficiently fat upper tails that cause the third term in the virtual valuation of equity auctions to dominate the second term. These results and the tail effects they demonstrate highlight the richness in the implications the formulation of optimal equity auctions can generate.

6 Conclusions

I analyze the impact of a pervasive, but little-studied, real-world feature of equity auctions: bidders usually differ ex ante in their characteristics, such as sizes or valuation distributions. The analysis applies to the sale of any non-divisible asset through equity payments, which encompasses a wide variety of economic situations.

I show bidder heterogeneity is important for equity auctions, and more so than for cash auctions. Central to this is the opacity of equity bids, the values of which depend on bidders’ private information. Bidder heterogeneity exacerbates this opacity, making the seller’s revenue sensitive to the auction format. I determine the revenue consequences of bidder heterogeneity for different equity-auction designs and derive the revenue-maximizing mechanism. I show how bidder heterogeneity alters the optimal design, and obtain the distinct implications of different sources of bidder heterogeneity.

My analysis reveals a number of unique features of the optimal design that have no analogue in the well-studied cash auctions. The most important of these features are (1) the preferential treatment of smaller bidders, (2) the opposing effects of the two forms of bidder heterogeneity on allocative efficiency, (3) the suboptimality of any payments by losing bidders, (4) the lower reserve prices relative to optimal cash auctions, and (5) the limiting behaviors when bidders are sufficiently large or small. These properties allow optimal equity auctions to respond to both sources of bidder heterogeneity and to maximally exploit the features of equities, generating revenues that always exceed optimal cash auctions, regardless of the nature of the bidder heterogeneity.

My paper focuses on equity auctions. Such auctions provide an ideal setting to study the effects of bidder heterogeneity because of their widespread use and the rele-
vance of bidder heterogeneity for their performance. More broadly, my work highlights the importance of accounting for bidder heterogeneity in securities auctions. Drawing from the insights of this paper’s analysis of equity auctions, bidder heterogeneity impacts the design and performance of securities auctions because securities’ values depend on bidders’ private information. The paper’s finding of how to optimally adjust the auction structure to restore the advantages of equities provides a stepping stone for future research of such adjustments for broader security classes.

References


7 Appendix A: Proofs (Under Revision)

Derivation of Equation (9): By equations (8) and (5), the following holds for all \( x_i \) and \( x'_i \):

\[
\begin{align*}
    u_i (x_i) - u_i (x'_i) &= [u_i (x_i) - v_i (x'_i, x_i)] + [v_i (x'_i, x_i) - u_i (x'_i)] \\
    &\leq u_i (x_i) - v_i (x'_i, x_i) \\
    &= (x_i - x'_i) q_i (x_i) G_i (x_i).
\end{align*}
\]

Thus

\[
    u_i (x'_i) \geq u_i (x_i) + (x'_i - x_i) q_i (x_i) G_i (x_i),
\]

which shows that at all \( x_i \), a line at \( x_i \) with a slope of \( q_i (x_i) G_i (x_i) \) supports the function \( u_i (\cdot) \). Thus \( u_i (\cdot) \) is convex. Because a convex function is absolutely continuous, it is differentiable almost everywhere in the interior of its domain. By equation (25), at every point that \( u_i (\cdot) \) is differentiable, \( \frac{du_i (x_i)}{dx_i} = q_i (x_i) G_i (x_i) \). Because every absolutely continuous function is the definite integral of its derivative, one has equation (9).

Proof of Theorem 1:

Lemma 5 The equity function \( q_i (\cdot) \) satisfies the following constraint:

\[
q_i (x_i) G_i (x_i) = \int_{x_i}^{x_i} \frac{u_i (x_i) + \omega_i (x)}{(V_i + x)^2} dx + \frac{V_i G_i (x_i)}{V_i + x_i} \frac{u_i (x_i) + \omega_i (x_i)}{V_i + x_i} + \int_{x_i}^{x_i} \frac{V_i G_i (x)}{(V_i + x)^2} dx.
\]

Rewrite (26) as \( q_i (x_i) G_i (x_i) \)

\[
\begin{align*}
    &= \frac{u_i (x_i)}{V_i + x_i} + \int_{x_i}^{x_i} \frac{\omega_i (t)}{(V_i + t)^2} dt + \frac{V_i G_i (x_i)}{V_i + x_i} \frac{u_i (x_i)}{V_i + x_i} + \int_{x_i}^{x_i} \frac{V_i G_i (t)}{(V_i + t)^2} dt \\
    &= \frac{V_i G_i (x_i)}{V_i + x_i} + \int_{x_i}^{x_i} \frac{V_i G_i (t)}{(V_i + t)^2} dt + \frac{u_i (x_i)}{V_i + x_i} + \delta_i (x_i),
\end{align*}
\]

where \( \delta_i (x_i) \equiv \frac{\omega_i (x_i)}{V_i + x_i} + \int_{x_i}^{x_i} \frac{\omega_i (t)}{(V_i + t)^2} dt \). Substituting equation (27) into equation (12)
Utilizing equation
\[ \pi_{s,i} \]
where
\[ Z \]
second term in equation (28) becomes
\[ y \]
Thus equation (28) becomes
\[ x_i = s_i + \frac{1}{V_i + x_i} \int_{\xi_i}^{x_i} \left[ x_i - \frac{(1 - F_i(x_i)) V_i}{V_i + x_i} \right] dx_i - u_i(x_i) \tau_i \]
\[ \]
where \( \tau_i \equiv \frac{1}{V_i + x_i} \int_{\xi_i}^{x_i} (1 - F_i(x_i)) dx_i + 1. \) Define \( L_i(x_i) = \int_{\xi_i}^{x_i} \frac{V_i G_i(t)}{(V_i + t)^2} dt \)
\[ \]
Utilizing equation (29), integrating by parts and noting \( H_i(\bar{x}_i) = L_i(\bar{x}_i) = 0, \) the second term in equation (28) becomes
\[ \]
Thus equation (28) becomes
\[ \]
where \( \phi_i(x_i) \) is given in equation (18). Substituting equation (3) into equation (30)
yields

\[ \pi_{s,i} = \int_{x_i} W_i(x) \phi_i(x_i) f(x) \, dx - u_i(x_i) \tau_i - \int_{x_i}^{x_i} (1 - F_i(x_i)) \delta_i(x_i) \, dx. \quad (31) \]

Substituting equation (31) into equation (11) and rearranging terms establishes the theorem.

**Proof of Lemma 1, Proposition 1, and Corollary 1:** Step 1: I first show Corollary 1. I start by showing truth telling is an equilibrium in this mechanism. For bidder \( i \) with valuation \( x_i \), compare its payoffs between reporting truthfully and over-reporting its value to be \( x_i' > x_i \). Three cases exist: (1) if truthful and over-reporting both result in winning, its payments are the same due to equations (22) and (23), and thus its payoffs are the same; (2) if truthful and over-reporting both result in losing, payoffs are zero in both situations; (3) if truthful reporting results in losing and over-reporting results in winning, \( \max_{j \neq i} \{ \phi_j(x_j) \} > \phi_i(x_i) \). Thus \( \phi_i^{-1}(\max_{j \neq i} \{ \phi_j(x_j) \}) > x_i \) and equation (23) shows \( y_i(x_{-i}) > x_i \), and equation (22) shows \( Q_i < \frac{V_i}{V_i + x_i} \). Its profit upon winning is negative; therefore over-reporting is not profitable. Summarizing these cases, truth telling weakly dominates over-reporting.

A similar argument shows truth telling also weakly dominates under-reporting.

Next, I show \( u_i(x_i) = 0 \) for all \( i \). Consider two situations: (1) \( G_i(x_i) = 0 \). Then equation (5) gives \( u_i(x_i) = 0 \). (2) \( G_i(x_i) > 0 \). Note equation (23) gives \( y_i(x_{-i}) \geq x_i \), which, by equation (22), gives \( q_i(x_i) \leq \frac{V_i}{V_i + x_i} \). Plugging this inequality into equation (2) yields \( u_i(x_i) \leq 0 \). On the other hand, a bidder can always ensure a zero profit by reporting a sufficiently low value, therefore its equilibrium expected profit must be nonnegative conditional on any valuation. Thus \( u_i(x_i) \geq 0 \), yielding \( u_i(x_i) = 0 \). Therefore \( \pi_{s,a} \) (equation 15) is zero. As cash payments are zero and losers do not pay, \( \pi_{s,b} \) (equation 16) is also zero. In addition, note the bidder with maximum \( \phi_i(x_i) \) wins provided \( \phi_i(x_i) > V_T \). By the arguments following Theorem 1, this mechanism maximizes \( \pi_{s,a} \), \( \pi_{s,b} \) and \( \pi_{s,c} \) simultaneously. Thus it maximizes \( \pi_s \) in equation (14), hence is optimal.

Step 2: I show the "if" part of Proposition 1 by assuming parts (i) through (iii) of the theorem. Then equations (4) and (21) yield

\[ q_i(x_i) G_i(x_i) = \frac{V_i G_i(x_i)}{V_i + x_i} + \int_{x_i}^{x_i} \frac{V_i G_i(t)}{(V_i + t)^2} \, dt. \quad (32) \]
Let \( v_i(x_i, z_i) \) denote \( i \)'s expected profit when \( i \) reports \( z_i \) while all others report truthfully; then (5) holds with \( \omega_i = 0 \) due to part \((ii)\) of the theorem. Thus 
\[
v_i(x_i, x_i) - v_i(x_i, z_i) = (V_i + x_i)(q_i(x_i) G_i(x_i) - q_i(z_i) G_i(z_i)) - V_i(G_i(x_i) - G_i(z_i)),
\]
which, upon plugging in (32), yields 
\[
v_i(x_i, x_i) - v_i(x_i, z_i) = V_i \left(G_i(x_i) - \frac{V_i + x_i}{V_i + z_i} G_i(z_i)\right) +
(V_i + x_i) \int_{z_i}^{x_i} \frac{V_i G_i(t)}{(V_i + t)^2} dt - V_i(G_i(x_i) - G_i(z_i))
\]
(33)
\[
= V_i \frac{z_i - x_i}{V_i + z_i} G_i(z_i) + (V_i + x_i) \int_{z_i}^{x_i} \frac{V_i G_i(t)}{(V_i + t)^2} dt.
\]
(34)

Further, part \((i)\) of the theorem shows \( W_i(\cdot, x_{-i}) \) is nondecreasing. Thus by (3), \( G_i(\cdot) \) is nondecreasing. From this, one can show
\[
\int_{z_i}^{x_i} \frac{V_i G_i(t)}{(V_i + t)^2} dt \geq \int_{z_i}^{x_i} \frac{V_i G_i(z_i)}{(V_i + t)^2} dt
\]
for both \( z_i > x_i \) and \( z_i < x_i \). Thus (34) yields
\[
v_i(x_i, x_i) - v_i(x_i, z_i) \geq V_i \frac{z_i - x_i}{V_i + z_i} G_i(z_i) + (V_i + x_i) \int_{z_i}^{x_i} \frac{V_i G_i(z_i)}{(V_i + t)^2} dt
\]
\[
= 0
\]
for all values of \( z_i \), establishing that truth-telling is an equilibrium.

Next, I show \( u_i(x_i) = 0 \) for all \( i \). Consider two situations: (1) \( G_i(x_i) = 0 \). Then equation (5) gives \( u_i(x_i) = 0 \). (2) \( G_i(x_i) > 0 \). Then equation (32) gives 
\[
q_i(x_i) = \frac{V_i}{V_i + x_i}.
\]
Thus equation (5) gives \( u_i(x_i) = 0 \). Therefore \( \pi_{s,a} \) (equation 15) is zero. Because losers do not pay, \( \pi_{s,b} \) (equation 16) is also zero. Part \((i)\) of the theorem shows the winner is the bidder with maximum \( \phi_i(x_i) \), provided that \( \phi_i(x_i) > V_T \). By the arguments following Theorem 1, \( \pi_{s,a}, \pi_{s,b}, \) and \( \pi_{s,c} \) simultaneously obtain their maximum values. Thus the mechanism is optimal.

Step 3: I prove Lemma 1 and the "only if" part of Proposition 1. As Corollary 1 shows, a mechanism exists that simultaneously maximizes \( \pi_{s,a}, \pi_{s,b}, \) and \( \pi_{s,c} \); an optimal mechanism cannot do worse, and therefore it must also simultaneously maximize \( \pi_{s,a}, \pi_{s,b}, \) and \( \pi_{s,c} \). By the arguments following Theorem 1, the lemma and part \((ii)\) of the theorem follow. The same arguments
also yield \( u_i(x_i) = 0 \) for all \( i \). Then, equating the right-hand sides of equations (27) and (4), and plugging in \( u_i(x_i) = 0 \), gives part (iii) of the theorem.

**Proofs of Corollaries 3 and 4:** Note that

\[
\int_{x_i}^{\bar{x}_i} (1 - F_i(t)) \, dt \leq (\bar{x}_i - x_i) (1 - F_i(x_i)),
\]

which yields

\[
\frac{V_i \int_{x_i}^{\bar{x}_i} (1 - F_i(t)) \, dt}{(V_i + x_i)^2 f_i(x_i)} \leq \frac{V_i (\bar{x}_i - x_i) (1 - F_i(x_i))}{(V_i + x_i)^2 f_i(x_i)}.
\]

Utilizing this relation, it is straightforward to establish the corollaries.

**Proof of Lemma 2:** Refer to equation (18). First note its first term \( x_i \) makes \( \phi_i \) increasing in \( x_i \). Next examine the second term: the numerator \( (1 - F_i(x_i)) \) is decreasing in \( x_i \), and the denominator \( (V_i + x_i)^2 f_i(x_i) \) is increasing in \( x_i \) under Assumption 2. Thus with its minus sign, the second term also makes \( \phi_i \) increasing in \( x_i \). Finally, examine the third term: again, the numerator \( \int_{x_i}^{\bar{x}_i} (1 - F_i(t)) \, dt \) is decreasing in \( x_i \), and the denominator \( (V_i + x_i)^2 f_i(x_i) \) is increasing in \( x_i \) under Assumption 2. Thus with its minus sign, the third term also makes \( \phi_i \) increasing in \( x_i \). Thus the lemma follows.

**Proof of Lemma 3:** Note

\[
\phi_i(x_i) - \psi_i(x_i) = \frac{x_i (1 - F_i(x_i))}{(V_i + x_i) f_i(x_i)} - \frac{V_i \int_{x_i}^{\bar{x}_i} (1 - F_i(t)) \, dt}{(V_i + x_i)^2 f_i(x_i)} \geq \frac{x_i (1 - F_i(x_i))}{(V_i + x_i) f_i(x_i)} - \frac{\int_{x_i}^{\bar{x}_i} (1 - F_i(t)) \, dt}{(V_i + x_i) f_i(x_i)} \geq \frac{(1 - F_i(x_i))}{(V_i + x_i) f_i(x_i)} (2x_i - \bar{x}_i),
\]

where equation (35) is used. Next, note that \( f_i(\cdot) \) is nondecreasing yields

\[
1 - F_i(x_i) \geq f_i(x_i) (\bar{x}_i - x_i).
\]

Thus the virtual valuation for cash auctions \( \psi_i(x_i) \leq 2x_i - \bar{x}_i \), and \( \psi_i(x_i) \geq V_T \) yields \( 2x_i \geq \bar{x}_i + V_T \). From equation (36), the lemma follows.

**Proof of Proposition 2:** Consider any realization of bidders’ valuations. Suppose the asset is sold in optimal cash auctions. Then, denote by \( i \) the bidder with the
highest virtual valuation for cash auctions, and denote by \( x_i \) its valuation, one then has \( \psi_i (x_i) \geq V_T \). Then by Lemma 3, \( \phi_i (x_i) > \psi_i (x_i) \). Therefore, \( \phi_i (x_i) > V_T \), implying the asset is also sold in optimal equity auctions under this realization of valuations. Thus the proposition holds. 

**Proof of Lemma 4:** Using equations (18) and (29), one has \( \frac{\partial \phi_i (x_i)}{\partial V_i} \) 

\[
\frac{\partial}{\partial V_i} \left( \frac{x_i}{V_i + x_i} - 1 \right) \frac{1 - F_i (x_i)}{f_i (x_i)} - \frac{\partial}{\partial V_i} \left[ \frac{V_i}{(V_i + x_i)^2} \right] \int_{x_i}^{\bar{x}_i} (1 - F_i (t)) \, dt \quad (38)
\]

\[
= - \frac{x_i}{(V_i + x_i)^2} \frac{1 - F_i (x_i)}{f_i (x_i)} - \frac{1}{(V_i + x_i)^2} \int_{x_i}^{\bar{x}_i} (1 - F_i (t)) \, dt + \frac{2V_i}{(V_i + x_i)^3} \int_{x_i}^{\bar{x}_i} (1 - F_i (t)) \, dt
\]

\[
= - \frac{x_i}{(V_i + x_i)^2} \frac{1 - F_i (x_i)}{f_i (x_i)} - \frac{x_i}{(V_i + x_i)^3} \frac{H_i (x_i)}{f_i (x_i)} + \frac{V_i}{(V_i + x_i)^3} \int_{x_i}^{\bar{x}_i} (1 - F_i (t)) \, dt
\]

\[
< - \frac{x_i}{(V_i + x_i)^2} \frac{1 - F_i (x_i)}{f_i (x_i)} + \frac{V_i}{(V_i + x_i)^3} \int_{x_i}^{\bar{x}_i} (1 - F_i (t)) \, dt.
\]

Consider two cases. (1) \( x_i \geq \frac{1}{2} \bar{x}_i \). Then equations (38) and (35) yield

\[
\frac{\partial \phi_i (x_i)}{\partial V_i} < - \frac{x_i}{(V_i + x_i)^2} \frac{1 - F_i (x_i)}{f_i (x_i)} + \frac{1}{(V_i + x_i)^2} \int_{x_i}^{\bar{x}_i} (1 - F_i (t)) \, dt
\]

\[
= \frac{1}{(V_i + x_i)^2} \frac{1 - F_i (x_i)}{f_i (x_i)} (\bar{x}_i - 2x_i),
\]

which establishes the lemma. (2) \( x_i < \frac{1}{2} \bar{x}_i \). Note the condition \( \phi_i (x_i) \geq V_T \) yields

\[
\frac{V_i \int_{x_i}^{\bar{x}_i} (1 - F_i (t)) \, dt}{(V_i + x_i)^2 f_i (x_i)} \leq x_i - \frac{(1 - F_i (x_i)) V_i}{(V_i + x_i) f_i (x_i)} - V_T. \quad (39)
\]

Substituting the above into equation (38) yields

\[
\frac{\partial \phi_i (x_i)}{\partial V_i} < - \frac{x_i}{(V_i + x_i)^2} \frac{1 - F_i (x_i)}{f_i (x_i)} + \frac{x_i - V_T}{V_i + x_i} - \frac{(1 - F_i (x_i)) V_i}{(V_i + x_i)^2 f_i (x_i)} \quad (40)
\]

\[
= - \frac{1}{V_i + x_i} \frac{1 - F_i (x_i)}{f_i (x_i)} + \frac{x_i - V_T}{V_i + x_i} \quad (41)
\]

\[
< \frac{1}{V_i + x_i} \left( x_i - \frac{1 - F_i (x_i)}{f_i (x_i)} \right), \quad (42)
\]

35
which, combined with (37), yields

\[
\frac{\partial \phi_i(x_i)}{\partial V_i} < \frac{1}{V_i + x_i} (x_i - (\bar{x}_i - x_i)) < 0.
\]

The lemma follows.

**Proof of Proposition 3.** Denote by \( \phi(V, x) \) the virtual valuation of a bidder who has market value \( V \), valuation \( x \), and the same valuation distribution as \( i \) and \( j \). Then

\[
\phi(V, x) = \phi(V_i, x) - \int_{V}^{V_i} \frac{\partial \phi(t, x)}{\partial t} dt.
\] (43)

Claim 1: If \( \phi(V_i, x) \geq V_T \), then \( \phi(V, x) \geq V_T \) for all \( V \in [V_j, V_i] \). Suppose that is not true; then some \( V \in [V_j, V_i] \) exists for which \( \phi(V, x) = V_T \). Denote by \( V^* \) the largest such value. On the other hand, from equation (43), Lemma 4 and the assumption \( \phi(V_i, x) \geq V_T \), there exists \( \epsilon > 0 \) such that \( \phi(V, x) > V_T \) for all \( V \in [V_i - \epsilon, V_i] \). Thus \( V^* < V_i \). Further, for any \( V \in (V^*, V_i) \), \( \phi(V, x) > V_T \). Then equation (43) and Lemma 4 show \( \phi(V^*, x) > V_T \), a contradiction. Thus claim 1 holds.

Claim 2: for all \( x \in [x_i, \bar{x}_i] \), if \( \phi_i(x_i) \equiv \phi(V_i, x_i) \geq V_T \), then \( \phi_j(x) \equiv \phi(V_j, x_i) \geq \phi_i(x) \). Claim 2 follows from Lemma 4, Claim 1, and equation (43).

Now consider bidders \( i \) and \( j \) with the same valuation \( x_i \). If \( \phi_i(x_i) < V_T \), then \( i \) has no chance of winning and the proposition trivially holds. Thus I assume \( \phi_i(x_i) \geq V_T \) without loss. The winning probability of bidder \( i \) is the probability that the following holds:

\[
\phi_i(x_i) > \max \left( \{ \phi_k(x_k) \}_{k \neq i,j}, \phi_j(x_i) \right),
\] (44)

where \( k \) denotes bidders other than \( i \) and \( j \). Similarly, the probability that bidder \( j \) will win is the probability that the following holds:

\[
\phi_j(x_i) > \max \left( \{ \phi_k(x_k) \}_{k \neq i,j}, \phi_i(x_i) \right).
\] (45)

Note the two bidders have the same valuation distributions. Claim 2 shows inequality (45) holds with a higher probability than (44), establishing the proposition.

**Proof of Proposition 4:**

If the asset is allocated in optimal equity auctions, denote by \( E \) the winning bidder. Similarly, if the asset is allocated in optimal cash auctions, denote by \( C \) the winning
bidder. I now prove part (i). Under uniform distributions, equation (18) becomes
\[
\phi_i (x_i) = x_i - \frac{V}{(V + x_i)} (\bar{x}_i - x_i) - \frac{1}{2} \frac{V}{(V + x_i)^2} (\bar{x}_i - x_i)^2,
\]
where \( V \) denotes bidders’ market value. The virtual valuation for cash auctions is \( \psi_i (x_i) = 2x_i - \bar{x}_i \). Consider two cases. Case 1: The asset is not sold in optimal cash auctions but is sold in optimal equity auctions. Then \( \phi_E (x_E) \geq V_T \) readily yields \( x_E \geq V_T \). Case 2: The asset is sold in optimal cash auctions. Then \( \psi_C (x_C) \geq V_T \), and Lemma 3 yields \( \phi_C (x_C) \geq \psi_C (x_C) \geq V_T \). Because \( \phi_E (x_E) \geq \phi_C (x_C) \), the asset is sold in optimal equity auctions. Next, I show \( x_C \leq x_E \) by the method of contradiction. Suppose instead \( x_C > x_E \), or \( \Delta > 0 \), where \( \Delta = x_C - x_E \). The winning criteria give \( 2x_C - \bar{x}_C \geq 2x_E - \bar{x}_E \), or
\[
\bar{x}_C - x_C \leq \bar{x}_E - x_E + \Delta,
\]
and
\[
x_E - \frac{V (\bar{x}_E - x_E)}{(V + x_E)} - \frac{1}{2} \frac{V (\bar{x}_E - x_E)^2}{(V + x_E)^2} - x_C + \frac{V (\bar{x}_C - x_C)}{(V + x_C)} + \frac{1}{2} \frac{V (\bar{x}_C - x_C)^2}{(V + x_C)^2} \geq 0.
\]
Because the asset is sold in optimal equity auctions, \( \phi_E (x_E) \geq V_T \) yields \( x_E > V_T \), hence one has \( x_C > x_E > V_T \). Utilizing this and equation (47), one has that the left hand side of equation (48)
\[
\leq \frac{V}{(V + x_C)} \Delta + \frac{1}{2} \frac{V [(\bar{x}_C - x_C)^2 - (\bar{x}_E - x_E)^2]}{(V + x_C)^2} - \Delta
\]
\[
= \frac{V}{(V + x_C)} \Delta + \frac{1}{2} \frac{V \Delta (\bar{x}_C - x_C + \bar{x}_E - x_E)}{(V + x_C)^2} - \Delta.
\]
Next, by assumption \( \Delta > 0 \) and equation (48), one can show \( \bar{x}_C - x_C > \bar{x}_E - x_E \). Thus, the second term in the right hand side of equation (49) satisfies
\[
\frac{1}{2} \frac{V \Delta (\bar{x}_C - x_C + \bar{x}_E - x_E)}{(V + x_C)^2} \leq \frac{V \Delta (\bar{x}_C - x_C)}{(V + x_C)^2}.
\]
Further, \( \psi_C(x_C) \geq V_T \) yields \( \bar{x}_C < 2x_C \). Thus
\[
\frac{1}{2} \frac{V \Delta (\bar{x}_C - x_C + \bar{x}_E - x_E)}{(V + x_C)^2} < \frac{V \Delta x_C}{(V + x_C)^2} < \frac{\Delta x_C}{V + x_C}.
\]

Plugging the above into (49), then one has that the left hand side of equation (48)
\[
< \frac{V}{V + x_C} \Delta + \frac{\Delta x_C}{V + x_C} - \Delta = - \frac{V_T}{V + x_C} \Delta
\]

Plugging the above into equation (48) yields \(- \frac{V_T}{V + x_C} \Delta > 0\), or \( \Delta < 0 \), contradicting the assumption that \( x_C > x_E \), hence proving \( x_C \leq x_E \). Thus the proposition is established for part (i).

I now prove part (ii) of the proposition. When \( k \) is sufficiently large, one has \( \phi_i(x_i) = x_i - \delta \frac{1 - F_i(x_i)}{f_i(x_i)} \), where \( \delta = \frac{V^*}{V^* + V^2} \in (0, 1) \). Again consider two cases. Case 1: The asset is not sold in optimal cash auctions but is sold in optimal equity auctions. Then one can readily show \( x_E \geq V_T \). Case 2: The asset is sold in optimal cash auctions. Then \( \psi_C(x_C) \geq V_T \). Because \( \phi_E(x_E) \geq \phi_C(x_C) \geq \psi_C(x_C) \geq V_T \), the asset is also sold in optimal equity auctions. Next I show \( x_C \leq x_E \). The winning criteria yield
\[
x_C = \frac{1 - F_C(x_C)}{f_C(x_C)} \geq x_E - \frac{1 - F_E(x_E)}{f_E(x_E)}
\]
and
\[
x_E - \delta \frac{1 - F_E(x_E)}{f_E(x_E)} \geq x_C - \delta \frac{1 - F_C(x_C)}{f_C(x_C)}.
\]
These two equations give
\[
\frac{1 - F_E(x_E)}{f_E(x_E)} - \frac{1 - F_C(x_C)}{f_C(x_C)} \geq x_E - x_C
\]
and
\[
\frac{1}{\delta} (x_E - x_C) \geq \frac{1 - F_E(x_E)}{f_E(x_E)} - \frac{1 - F_C(x_C)}{f_C(x_C)}.
\]
Thus $\frac{1}{5} (x_E - x_C) \geq x_E - x_C$, yielding $\left(\frac{1}{5} - 1\right) (x_E - x_C) \geq 0$. Because $\left(\frac{1}{5} - 1\right) > 0$, $x_E \geq x_C$. Thus part (ii) of the proposition is established. Part (iii) of the proposition follows similarly by referring to Corollary 4.

**Proof of Proposition 5:** Consider the generalized framework in section 8.1, where bidders can optionally pay cash. Note Theorem 1 holds in this framework with equation (50) replacing $\omega_i$ in equation (16). Using the same arguments as in the proof of Proposition 6, and noting the term $\pi_{s,b}$ is strictly negative in any cash auction, the proposition follows (see section 8.1 for discussions of technical subtleties to account for in constructing the proof).  

$\blacksquare$
8 Appendix B: Extensions with Cash Payments

In the main model, bidders pay with equity only. In this section, I extend the model by incorporating cash payments in two ways: section 8.1 considers optional cash payments and section 8.2 considers mandatory cash payments.

8.1 Optimal Mechanisms When Bidders May Optionally Offer Cash

In this part I assume all bidders (winner and losers) can pay with any combination of cash and equity, where the cash payment is nonnegative. I show it is not optimal for the seller to accept cash payments; thus the optimal mechanism is the same as in the main model.

The analysis follows the same general procedures as in the main model. In the main model, the direct mechanism is represented by \((W, Q)\), here I augment this representation to \((W, Q, M)\), where \(M : \chi \times B \times A \rightarrow \mathbb{R}^n_{\geq 0}\) is the cash-payment rule: \(M_i(x, j)\) is the cash amount bidder \(i\) pays when bidders report \(x\) and bidder \(j\) wins.

Then equation (2) becomes
\[
v_i(x_i, z_i) = \int_{x_{-i}} [(V_i + x_i - M_i(z_i, x_{-i}, i)) Q_i(z_i, x_{-i}, i) - V_i] W_i(z_i, x_{-i}) f_{-i}(x_{-i}) d x_{-i}
+ \sum_{j \neq i} \int_{x_{-i}} [(V_i - M_i(z_i, x_{-i}, j)) Q_i(z_i, x_{-i}, j) - V_i] W_j(z_i, x_{-i}) f_{-i}(x_{-i}) d x_{-i},
\]
and equation (6) becomes
\[
\omega_i(z_i) = \int_{x_{-i}} M_i(z_i, x_{-i}, i) Q_i(z_i, x_{-i}, i) W_i(z_i, x_{-i}) f_{-i}(x_{-i}) d x_{-i} + \sum_{j \neq i} \int_{x_{-i}} [V_i (1 - Q_i(z_i, x_{-i}, j)) + M_i(z_i, x_{-i}, j) Q_i(z_i, x_{-i}, j)] W_j(z_i, x_{-i}) f_{-i}(x_{-i}) d x (50)
\]

Following the same procedures as in the main analysis, one can show Theorem 1 holds with equation (50) replacing \(\omega_i\) in equation (16). I show the following:
Proposition 6  In the optimal mechanism, bidders pay with equity only with no cash component. This result holds independently of any ex-ante bidder heterogeneity.

The no-cash result in Proposition 6 can be understood on multiple levels. On an intuitive level, a bidder’s cash payment would result in the bidder demanding a higher equity share in compensation upon winning, thereby increasing the bidders’ rents and reducing the seller’s revenue. More specifically, this intuition is reflected in equation (26), which suggests that if one reduces cash transfers while keeping the allocation $W$ unchanged and reducing $q_i(\cdot)$ accordingly (to satisfy (26), which is a necessary condition for incentive compatibility), the expected revenue would increase, as (12) implies.

On a technical level, however, the arguments above do not suffice as a proof, because cash and equity payments interact with the incentive compatibility constraint in a complicated manner. To see this complication, suppose the non-negativity constraint does not bind in some incentive-compatible mechanism and one adopts the procedure of reducing cash transfers while decreasing $q_i$ to satisfy (26). For this procedure to be valid, one would need to show the new $q_i$ is implementable; i.e., the corresponding $Q$ exists such that the system of equations (1), (4), (6), and (26) hold. Furthermore, equation (26) is only a necessary condition for incentive compatibility; one would need to show the new $q_i$ satisfies any additional conditions imposed by the incentive compatibility constraint. In particular, the fact that $u_i(\cdot)$ in equation (9) is convex requires $q_i(\cdot)G_i(\cdot)$ to be nondecreasing.

Indeed, the no-cash result is a consequence of the fact that Theorem 1 holds with equation (50) replacing $\omega_i$ in equation (16). The validity of Theorem 1 under such replacement shows the effects of cash payments on expected revenue are separable from other sources. Combined with the fact that an incentive-compatible mechanism exists that simultaneously maximizes the contributions to the expected revenue from each source, and that positive cash transfers strictly lower the part of contributions from cash payments, it follows that any amounts of cash payments are suboptimal.

8.2 Optimal Mechanisms When Some Bidders Must Offer Cash

In this part, I extend the main model in another direction and derive the optimal mechanism in settings in which some bidders must offer cash. Specifically, let
$l \in \{1, 2, ..., n - 1\}$; assume bidders $i \leq l$ must offer cash, whereas bidders $i > l$ offer equities. Assume $\psi_i(\cdot)$ is increasing for all $i \leq l$ and $\phi_i(\cdot)$ is increasing for all $i > l$.

**Proposition 7** Mechanism $(W, M, Q)$ is optimal if and only if

(i) for all $i$, the winning rule is

$$W_i(x) = \begin{cases} 1 & \text{if } \kappa_i(x_i) > \max_{j \neq i} \{\kappa_j(x_j)\} \text{ and } \kappa_i(x_i) \geq V_T \\ 0 & \text{if } \kappa_i(x_i) < \max_{j \neq i} \{\kappa_j(x_j)\} \text{ or } \kappa_i(x_i) < V_T \end{cases}$$

where, for all $j$ and $x_j$, $\kappa_j(x_j) \equiv \psi_j(x_j)$ if $j \leq l$ and $\kappa_j(x_j) \equiv \phi_j(x_j)$ if $j > l$.

(ii) For all $i \leq l$ and $x_i$, the cash-payment rule satisfies

$$\int_{\chi_{-i}} \sum_{j=1}^{n} M_i(x_i, x_{-i}, j) W_j(x_i, x_{-i}) f_{-i}(x_{-i}) \, dx_{-i} = (V_i + x_i) G_i(x_i) - \int_{\xi_i}^{x_i} G_i(t) \, dt.$$  

(iii) For all bidders $i > l$, the bidder does not pay upon losing, and the equity-retention rule upon winning satisfies part (iii) of Proposition 1.

Note that the cash-payment rule in equation (52) corresponds to the equity-retention rule in equation (21) under the limit $V_i \to \infty$. To see this correspondence, suppose $i$ offers equity. Then the cash value of the equity it pays upon winning is $(1 - Q_i(x_i, x_{-i}, i))(V_i + x_i)$. From equation (21), the expected cash value of its equity payment conditional on $x_i$ is

$$(V_i + x_i) \int_{\chi_{-i}} (1 - Q_i(x_i, x_{-i}, i)) W_i(x_i, x_{-i}) f_{-i}(x_{-i}) \, dx_{-i}$$

$$= (V_i + x_i) \left( G_i(x_i) - \frac{V_i G_i(x_i)}{V_i + x_i} + \int_{x_i}^{x_i} \frac{V_i G_i(t)}{(V_i + t)^2} \, dt \right).$$

Upon taking the limit $V_i \to \infty$, the above becomes $(V_i + x_i) G_i(x_i) - \int_{\xi_i}^{x_i} G_i(t) \, dt$, which is precisely the right-hand side of equation (52).

9 Appendix C: Generalization Outside of Equities

In this section, I generalize the analysis to a larger class of securities than the class of equities. I show virtual valuations can also be defined for this larger class of securities, and via these valuations, the optimal mechanism can be formulated.
I extend the main model in two ways: I let bidders submit bids from sets of securities formed by linear combinations of two given securities, and I let a bidder’s realized valuation, conditional on the bidder’s private type, be stochastic. I retain all structure and notation of the main model otherwise.

Let \( S(z) \) describe a security, which is the payment to the seller when the merged firm realizes cash flow \( z \). For each bidder \( i \), let \( S^*_1(\cdot) \) and \( S^*_2(\cdot) \) be two base securities. Bidder \( i \) submits bids from the following set of securities formed by all linear combinations of the base securities:

\[
S_{ri}(z) = (1-r)S^*_1(z) + rS^*_2(z) \quad \text{for all } z, 
\]

where \( r \) indexes the securities. Such a set of securities (in (53)) falls into the convex set of securities introduced in DeMarzo, Kremer, and Skrzypacz (2005), which is equal to its convex hull. Note that \( S^*_1(\cdot) \) and \( S^*_2(\cdot) \) are bidder-specific, i.e., different bidders can submit bids from different sets of securities.

For each bidder \( i \) with private type \( x_i \), let random variable \( Z_{x_i} \) denote the actual valuation it can realize. Without loss of generality, define the expected value of \( Z_{x_i} \) to be \( x_i \), i.e., \( x_i = \mathbb{E}[Z_{x_i}] \) for all \( x_i \in [\underline{x}_i, \bar{x}_i] \). For all bidders \( i \) and base securities \( j \in \{1, 2\} \), define

\[
K^*_j(x_i) \equiv \mathbb{E}[S^*_j(V_i + Z_{x_i}) | x_i],
\]

which is the expected payment to the seller when bidder \( i \), whose private type is \( x_i \), pays security \( S^*_j \). Similarly, for all \( i \) and \( r \), define

\[
K_{ri}(x_i) \equiv \mathbb{E}[S_{ri}(V_i + Z_{x_i}) | x_i],
\]

which is the expected payment to the seller when bidder \( i \) with private type of \( x_i \) pays security \( S_{ri} \). From equations (53) and (54), one has

\[
K_{ri}(x_i) = (1-r)K^*_1(x_i) + rK^*_2(x_i)
\]

\[
= K^*_1(x_i) + rD^*_i(x_i),
\]

for all \( r, i, x_i \), where

\[
D^*_i(x_i) \equiv K^*_2(x_i) - K^*_1(x_i).
\]

\(^{20}\)With equity bids, allowing for stochastic synergies has no impact, because values of equities depend only on the expected value of the synergy, not on the details of the distribution.
I require $K^*_1(x_i)$ and $K^*_2(x_i)$ to be continuous and differentiable, and $D^*_i(x_i) > 0$ for all $i$ and $x_i$. These requirements correspond to mild restrictions on $S^*_1(\cdot)$, $S^*_2(\cdot)$, and the distributions of $Z_{x_i}$.\footnote{Additional restrictions on $S^*_1(\cdot)$, $S^*_2(\cdot)$, and $r$ are necessary to ensure the feasibility of the securities (for details on the feasibility of securities, see DeMarzo, Kremer, and Skrzypacz 2005). For generality of the formulation of the virtual valuation, I do not impose such restrictions.} I assume for simplicity that losing bidders do not pay. This assumption is without loss of generality—as in the earlier analysis, one can show that losers do not pay in the optimal mechanism in security-bid auctions under general conditions.

**Analysis.** Consider a direct mechanism $(W, R)$ in which a truth-telling equilibrium exists and (1) $W: x \to \mathbb{R}_{\geq 0}^n$ is the winning rule, where $\Sigma_{i=1}^n W_i(x) \leq 1$ for all $x$ and $W_i(x)$ is bidder $i$’s winning probability when bidders report valuations $x$; and (2) $R: x \to \mathbb{R}_{\geq 0}^n$ is the payment rule: $R_i(x)$ is the index (i.e., the $r$) of the security bidder $i$ pays upon winning when bidders report $x$.

Denote bidder $i$’s expected profit when it has $x_i$ but reports $z_i$, and all other bidders report truthfully by $v_i(x_i, z_i)$; then

$$v_i(x_i, z_i) = \int_{x_{-i}} [x_i - (K^*_1(x_i) + R_i(z_i, x_{-i}) D^*_i(x_i))] W_i(z_i, x_{-i}) f_{-i}(x_{-i}) \, dx_{-i}. \quad (55)$$

To express (55) more concisely, denote the expected value of $R_i$ that bidder $i$ pays upon winning if it reports a value $z_i$ and all others report truthfully by $r_i(z_i)$:

$$r_i(z_i) G_i(z_i) = \int_{x_{-i}} R_i(z_i, x_{-i}) W_i(z_i, x_{-i}) f_{-i}(x_{-i}) \, dx_{-i}. \quad (56)$$

Using equations (3) and (56), bidder $i$’s expected profit is

$$v_i(x_i, z_i) = [x_i - (K^*_1(x_i) + r_i(z_i) D^*_i(x_i))] G_i(z_i), \quad (57)$$

Denoting bidder $i$’s equilibrium expected profit by

$$u_i(x_i) \equiv v_i(x_i, x_i) = [x_i - (K^*_1(x_i) + r_i(x_i) D^*_i(x_i))] G_i(x_i), \quad (58)$$

$$\int_{x_{-i}} [x_i - (K^*_1(x_i) + r_i(z_i) D^*_i(x_i))] W_i(z_i, x_{-i}) f_{-i}(x_{-i}) \, dx_{-i}. \quad (55)$$

To express (55) more concisely, denote the expected value of $R_i$ that bidder $i$ pays upon winning if it reports a value $z_i$ and all others report truthfully by $r_i(z_i)$:

$$r_i(z_i) G_i(z_i) = \int_{x_{-i}} R_i(z_i, x_{-i}) W_i(z_i, x_{-i}) f_{-i}(x_{-i}) \, dx_{-i}. \quad (56)$$

Using equations (3) and (56), bidder $i$’s expected profit is

$$v_i(x_i, z_i) = [x_i - (K^*_1(x_i) + r_i(z_i) D^*_i(x_i))] G_i(z_i), \quad (57)$$

Denoting bidder $i$’s equilibrium expected profit by

$$u_i(x_i) \equiv v_i(x_i, x_i) = [x_i - (K^*_1(x_i) + r_i(x_i) D^*_i(x_i))] G_i(x_i), \quad (58)$$
incentive compatibility conditions yield:

\[ u_i(x_i) = u_i(x_i) + \int_0^{x_i} (1 - K_{1i}^*(t) - r_i(t)D_i^{**}(t))G_i(t)\, dt. \]  

(59)

Using similar techniques as in the preceding analysis, I show the seller’s revenue decomposes and virtual valuations exist in ways analogous to equity auctions:

**Proposition 8 (Generalized virtual valuation):** The seller’s expected revenue is

\[ \pi_s = \pi_{s,a} + \pi_{s,c}, \]  

(60)

where

\[ \pi_{s,a} \equiv -\sum_{i=1}^n u_i(x_i) \left( \frac{1}{D_i'(x_i)} \int_0^{x_i} (1 - F_i(x_i)) D_i^{**}(x_i)\, dx_i + 1 \right), \]  

(61)

\[ \pi_{s,c} \equiv \int \left[ \sum_{i=1}^n \left( W_i(x) \left( \hat{\phi}_i(x_i) - V_T \right) \right) f(x)\, dx + V_T \right], \]  

(62)

and \( \hat{\phi}_i(x_i) \) is the virtual valuation in equation (63).

**Definition 5** The virtual valuation for the set of securities in (53) is

\[ \hat{\phi}_i(x_i) = x_i - \frac{1 - F_i(x_i)}{f_i(x_i)} \left[ 1 - K_{1i}^*(x_i) - D_i^{**}(x_i) \frac{x_i - K_{1i}^*(x_i)}{D_i^*(x_i)} \right] - \int_0^{x_i} \frac{(1 - F_i(t))D_i^{**}(t)\, dt}{f_i(x_i)} \left[ 1 - K_{1i}^*(x_i) - D_i^{**}(x_i) \frac{D_i^{**}(x_i)}{D_i^*(x_i)} \left[ x_i - K_{1i}^*(x_i) \right] \right]. \]  

(63)

Proposition 8 shows that virtual valuation exists for the class of securities in (53), which is larger than the class of equities. As in equity auctions, the optimal mechanism can be constructed via the virtual valuation, analogous to Theorem 1 and Corollary 1. Because the class of securities in (53) can differ for each bidder, the optimal mechanism has a salient feature that accommodates situations in which different bidders offer different types of securities.

The virtual valuation in equation (63) depends on the form of \( K_{1i}^*(\cdot) \) and \( D_i^*(\cdot) \), which depends on the base securities \( S_{1i}^*(\cdot) \) and \( S_{2i}^*(\cdot) \) and the conditional distribution of valuation \( Z_{xi} \). Note that the virtual valuation is a property for the set of the securities. As the set in (53) is spanned by linear combinations of the base securities \( S_{1i}^*(\cdot) \) and \( S_{2i}^*(\cdot) \), the set is invariant if the base securities themselves undergo similar
linear transformations. Indeed, one can verify that the virtual valuation is invariant to such transformations.

The virtual valuation in equation (63) entails a rich structure. Although a detailed examination is outside the scope of this paper, note that equation (63) encompasses the virtual valuations for equity and for cash auctions as special cases. To see this, let \( S_{1i}^* (\cdot) \) be the null-security, which pays off zero for any cash flow realizations. Then \( K_{1i}^* (x_i) = 0 \) for all \( x_i \). If \( S_{2i}^* \) is the full equity, i.e., \( S_{2i}^* (z) = z \), then \( D_i^* (x_i) = K_{2i}^* (x_i) = V_i + x_i \). It is straightforward to show equation (63) reduces to (18), the virtual valuation for equity auctions. If, instead, \( S_{2i}^* (z) = 1 + \alpha (z - V_i) \), where \( \alpha > 0 \), then \( D_i^* (x_i) = K_{2i}^* (x_i) = 1 + \alpha (x_i - V_i) \). One can show in the limit when \( \alpha \) goes to zero, equation (63) reduces to (19), the virtual valuation for cash auctions.

10 Appendix D: Further Details

Section 10.2 provides further details on the abnormalities discussed in section 5.1. Section 10.2 presents an alternative interpretation of the virtual valuation. Section 10.3 describes standard second-price equity auctions. Section 10.4 provides numerical results on the comparisons of optimal equity and optimal cash auctions. Section 10.5 contains proofs for results in Appendices B through D.

10.1 More on Abnormalities

As discussed in the main text, the relative importance of the third term in the virtual valuation for equity auctions increases in the size of the upper tails of the valuation distribution. With sufficiently high tails, the third term can dominate over the second term and lead to counter-intuitive results, as I show in the example below.

**Example 3** (Abnormalities): Suppose bidder 1 has valuation distribution \( f_1 (x_1) = \frac{A}{(x_1 - B)^2} \) over \([\underline{x}_1, \overline{x}_1] \), where \( B < \underline{x}_1 < \overline{x}_1 \) and \( A \equiv \frac{(\overline{x}_1 - B)(\overline{x}_1 - B)}{\overline{x}_1 - \underline{x}_1} \) normalizes the distribution.
Direct calculation yields $1 - F_1(s_1) = \frac{A}{x_1 - B} - \frac{A}{\bar{x}_1 - B}$. Then the virtual valuations for equity and cash auctions are

$$\phi_1(s_1) = s_1 - \frac{V_1}{V_1 + x_1} \left( x_1 - B - \frac{(x_1 - B)^2}{\bar{x}_1 - B} \right) - \frac{V_1 (x_1 - B)^2}{(V_1 + x_1)^2} \left( \ln \frac{\bar{x}_1 - B}{x_1 - B} - \frac{\bar{x}_1 - x_1}{\bar{x}_1 - B} \right)$$

and $\psi_1(s_1) = B + \frac{(x_1 - B)^2}{\bar{x}_1 - B}$, respectively. Inspection of equation (64) shows that when $\bar{x}_1$ (which measures the extent of the tails) increases arbitrarily, the third term goes to infinity, dominating the second term, which stays finite. This dominance suggests counter-intuitive results may obtain when $\bar{x}_1$ is sufficiently high.

To establish claims 1 and 2 in section 5.1, let $V_1 = 10$, $V_T = 1$, $B = 1.1$, $\bar{x}_1 = 4$, and $\bar{s} = 101$. One can verify both $\phi_1(\cdot)$ and $\psi_1(\cdot)$ are strictly increasing on $[\bar{x}_1, \bar{x}_1]$. At $x_1^* = 6$, $\psi_1(x_1^*) = 0.34$, which exceeds $\phi_1(x_1^*) = 0.15$, demonstrating result 1. Further, suppose bidder 1 is the single bidder (or suppose all other bidders have the same market value and valuation distribution as bidder 1). Because $\psi_1(4) = 0.18 > 0$, the asset is always sold in optimal cash auctions. However, because $\phi_1(4) = -0.11 < 0$, the asset is not always sold in optimal equity auctions. Indeed, because $\phi_1^{-1}(1) = 5.28$, the asset is sold only with probability $1 - F_1(5.28) = 68\%$ (when bidder 1 is the single bidder), yielding result 2. To establish claim 3, let $V_1 = 50$, $V_T = 1$, $B = 1.1$, $\bar{x}_1 = 6$, and $\bar{x}_1 = 101$. One can verify $\phi_1(\cdot)$ is strictly increasing on $[\bar{x}_1, \bar{x}_1]$. At $x_1^* = 6$, $\frac{\partial \phi_1(x_1^*)}{\partial V_1} = 0.0035 > 0$ and $\phi_1(x_1^*) = 0.05 > 0$, demonstrating result 3.

The fat upper tails of the valuation distribution drive these seemingly counter-intuitive results. To understand why optimal equity auctions may result in lower probabilities of trade than optimal cash auctions, consider a single-bidder setting. In both cash and equity auctions, raising the reserve price has two opposing effects: an increase in the expected payments conditional on a trade, and a reduction in the probability of trade. In equity auctions, a higher reserve price leads to a higher fraction of equity offer. Because the value of equity is tied to a bidder’s actual valuation, the gains from raising the reserve price increases in the extent of the high tails in the valuation distribution. However, in cash auctions, such an increase does not arise, because the value of a cash payment does not depend on a bidder’s valuation. Therefore, with sufficiently fat upper tails, the gains from raising the reserve price become more significant in equity auctions than in cash auctions. This encourages the seller to set a higher reserve price in optimal equity auctions, resulting in a lower probability
of trade, contrary to the higher probability of trade that results whenever the upper
tail is not too fat (Proposition 2).

Similarly, the counter-intuitive result that larger bidders in optimal equity auc-
tions may be more likely to win (part \((iii)\)) is driven by the fact that the high-tail
effects in equity auctions become more significant as a bidder’s market value decreases.
Specifically, consider two bidders with the same valuation distribution and valuation,
but different market values. Having the larger bidder win does have a disadvantage
that the seller can extract fewer rents when the bidders have the same given valua-
tion. However, having the larger bidder win also has an effect of essentially setting
a higher "reserve price" for the smaller bidder. As discussed earlier, a higher reserve
price corresponds to a higher equity fraction that the bidder pays, and the value of an
equity payment increases in the extent of the valuation tails. Because the tail-effects
are more pronounced for the smaller than for the larger bidder, when the tails are
sufficiently fat, setting a higher reserve price for the smaller bidder allows the seller
to extract more rents, when bidders have higher valuations. Balancing this trade-offs
in the rents extracted at the given valuation (or in its neighborhood) versus at higher
valuation, with sufficiently fat tails in the valuation distribution, the latter effect
dominates, thereby making it optimal to award larger instead of smaller bidders.

These counter-intuitive results and the tail effects they demonstrate offer further
insights into the differences between equity and cash bids at a more subtle level, high-
lighting the richness in the implications the formulation of optimal equity auctions
can generate.

10.2 Alternative Interpretation

Given the result that virtual valuation for equity auctions exists, as Theorem 1 estab-
lishes, the virtual valuation can be alternatively interpreted as the marginal revenue
in a monopolist pricing situation, the same as what Bulow and Roberts (1989) have
shown in cash auctions.

Specifically, consider bidder \(i\) in isolation. Suppose the seller makes a take-it-or-
leave-it offer that makes the bidder indifferent between accepting and rejecting if it
has a valuation \(x_i\); then, the seller demands an equity fraction of \(\frac{x_i}{V_i + x_i}\). The proba-
bility the bidder accepts the offer is \(p(x_i) = 1 - F_i(x_i)\), which can be interpreted as
the quantity demanded by the bidder. Thus, the seller’s revenue is

\[ \Pi = (1 - F_i(x_i)) \left( \frac{x_i}{V_i + x_i} (V_i + E[x|x \geq x_i]) - V_T \right) \]

\[ = (1 - F_i(x_i)) \left( x_i + \frac{x_i}{V_i + x_i} E[x - x_i|x \geq x_i] \right). \]

Taking derivative with respect to \( x_i \) yields

\[ \frac{d\Pi}{dx_i} = -f_i(x_i)x_i + \frac{V_i (1 - F_i(x_i))}{V_i + x_i} + \left[ \frac{d}{dx_i} \left( \frac{x_i}{V_i + x_i} \right) \right] (1 - F_i(x_i)) E[x - x_i|x \geq x_i]. \] (65)

Note \( (1 - F_i(x_i))E[x - x_i|x \geq x_i] = \int_{x_i}^{x_i} (x - x_i) f_i(x) dx = -\int_{x_i}^{\infty} (x - x_i) d(1 - F_i(x)) = \int_{x_i}^{\infty} (1 - F_i(x)) dx \). Further, \( \frac{dp}{dx_i} = -f_i(x_i) \) or \( \frac{dx_i}{dp} = -\frac{1}{f_i(x_i)} \). The marginal revenue is \( \frac{d\Pi}{dp} = \frac{d\Pi}{dx_i} \frac{dx_i}{dp} \), which, upon straightforward algebra, takes the same form as the virtual valuation.

Note the third term in the virtual valuation can be expressed as

\[ \frac{dx_i}{dp} \left[ \frac{d}{dx_i} \left( \frac{x_i}{V_i + x_i} \right) \right] (1 - F_i(x_i)) E[x - x_i|x \geq x_i]. \] This expression reveals that (1) the third term relates to the rate of change in the equity fraction the bidder offers, as the term \( \frac{d}{dx_i} \left( \frac{x_i}{V_i + x_i} \right) \) shows, and (2) the third term manifests the tail effects of the valuation distribution, as the term \( E[x - x_i|x \geq x_i] \) shows.

### 10.3 Standard Second-Price Equity Auctions

This part describes standard second-price equity auctions, in which I impose discriminatory reserve prices to enhance the seller’s expected revenue.

- **Second-price auction.** Each bidder \( i \) offers the seller a fraction \( p_i \) of equity. Let \( p_i \) be the highest \( p \)-offer and \( \{r_i\}_{i=1}^n \) be the reserve prices. If \( p_i \geq r_i \), bidder \( i \) wins and pays an equity fraction \( \max\{r_i, \max_{j \neq i} \{p_j\}\} \); if \( p_i < r_i \), no bidder wins. The bidding strategy is truthful:

\[ p_i(x_i) = \frac{x_i}{V_i + x_i}. \] (66)

Utilizing Theorem 1, the following lemma derives the optimal reserve prices that maximize the seller’s expected revenue.
Lemma 6 The following reserve prices are optimal in maximizing the seller’s expected revenue in the second-price auction:

\[ r_i = \frac{\phi_i^{-1}(V_T)}{V_i + \phi_i^{-1}(V_T)}. \]

10.4 Comparing Optimal Equity and Optimal Cash Auctions

This section presents numerical illustrations and comparisons of optimal equity auctions and optimal cash auctions to demonstrate Proposition 4 and the related results. Figure 1 concerns a two-bidder situation in which both bidders and the target have a market value of 3. Bidder 1’s synergy is uniformly distributed over \([1, 2]\). Bidder 2’s synergy is uniformly distributed over \([1.5 - \frac{w}{2}, 1.5 + \frac{w}{2}]\), which has the same mean as bidder 1’s distribution but a different width \(w\). The left panel plots the expected revenue difference between optimal equity and optimal cash auctions. The results confirm optimal equity auctions generate strictly higher expected revenues for all values of bidder 2’s synergy distribution width, reflecting the fact that the optimal mechanism properly accounts for both sources of ex-ante bidder heterogeneity and maximally takes the advantages of equity over cash bids.

The results also show the revenue difference increases in bidder 2’s synergy distribution width. Intuitively, the revenue difference between optimal equity and optimal cash auctions represents the reduction in bidders’ rents due to the optimal use of equities. When bidders’ synergy distribution widths increase, the seller’s knowledge of the bidders’ synergies becomes less precise, and bidders earn more rents accordingly. Consequently, the amount of reduction in bidders’ rents due to the optimal use of equities also increases.

The right panel of the figure plots the ratio of bidder 1’s winning probability over bidder 2’s. The results show both ratios increase in bidder 2’s distribution width \(w\): they are less than 1 at \(w < 1\), equal 1 at \(w = 1\), and exceed 1 at \(w > 1\). Note an efficient mechanism would result in a ratio of 1 for all values of \(w\); thus a deviation of the ratio from 1 represents a degree of inefficiency. The figure demonstrates two important features. First, these results highlight the extent to which the allocation of optimal equity auctions favors the bidder with less dispersed synergy. Intuitively, such bidder represents an outside option, allowing the seller to demand higher bids from the other bidder. Second, as the figure demonstrates, the ratio for optimal eq-
Figure 1: Comparisons of optimal equity and optimal cash auctions under heterogeneity in bidders’ synergy distributions. The left panel plots the expected revenue difference between optimal equity and optimal cash auctions. The right plots the ratio of bidder 1’s winning probability over bidder 2’s, in which the solid line with points (red) is for optimal equity auctions and dash dots with hexagram (black) are for optimal cash auctions.

The ratio of bidder 1’s winning probability over bidder 2’s is closer to 1 than the ratio for optimal cash auctions at all values of $w$, suggesting optimal equity auctions are more efficient than optimal cash auctions, consistent with Proposition 4. Intuitively, the seller can extract more rents in equity auctions than in cash auctions, making a bidder’s virtual valuation closer to its actual synergy. This effect reduces the differences between bidders’ virtual valuations due to their differing synergy distributions, thereby resulting in more efficient allocations.

10.5 Proofs for Appendices B through D

Proof of Proposition 6: Note Theorem 1 holds with equation (50) replacing $\omega_i$ in equation (16). Note also that optimal equity auctions are optimal in this framework because they simultaneously maximize $\pi_{s,a}$, $\pi_{s,b}$, and $\pi_{s,c}$ (in their generalized forms), and no other mechanism can do better than that. Because $\pi_{s,b}$ is zero in optimal equity auctions but is strictly negative with any cash transfer, by the arguments following Theorem 1, the lemma follows.

Proof of Proposition 7: For bidder $i$ who offers cash, plug $g_i(x_i) = 1$ into equation
and then equation (10). Following similar steps as in the proofs of Theorem 1 and Proposition (1) establishes the proposition.

**Proof of Lemma 6:** Note reserve prices do not affect the bidding strategies and the term \( \pi_{s,b} \) in equation (14) is zero. Note also that at the reserve prices given in the lemma, \( u_i (x_i) = 0 \) for all \( i \) and thus \( \pi_{s,a} = 0 \), which obtains its maximum value. Now I examine how reserve prices affect \( \pi_{s,c} \). Consider any realization of valuations \( x \) and assume that bidder \( k \) with valuation \( x_k \) has the highest bid. Given the reserve price in the lemma, the asset is allocated if and only if

\[
\frac{x_k}{V_k + x_k} \geq \frac{\phi_k^{-1}(V_T)}{V_k + \phi_k^{-1}(V_T)}.
\]

As \( \frac{x_k}{V_k + x_k} \) strictly increases in \( x_k \), the asset is allocated if and only if \( x_k \geq \phi_k^{-1}(V_T) \). Referring to equation (17), one then has

\[
\sum_{i=1}^{n} W_i (x) \phi_i (x_i) = \begin{cases} 
\phi_k (x_k) & \text{if } x_k \geq \phi_k^{-1}(V_T) \\
0 & \text{if } x_k < \phi_k^{-1}(V_T)
\end{cases}.
\]

The right-hand side of equation (67) is no less than would be obtained by any other reserve price, given any value of \( x_k \) (recall the design problem is regular as has been assumed earlier). Thus the lemma is established.