Early Default Risk and Surrender Risk: Impacts on Participating Life Insurance Policies

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November 2016

Abstract

We study the risk-neutral valuation of participating life insurance policies with surrender guarantees when an early default mechanism, forcing an insurance company to be liquidated once a solvency threshold is reached before maturity, is imposed by a regulator. The early default regulation affects the policies’ value not only directly via changing policies’ payment streams but also indirectly via influencing policyholders’ surrender behavior. In this paper, we endogenize surrender risk by assuming the surrender intensity of a representative policyholder bounded from below and from above, and uncover impacts of the regulation on the policyholder’s surrender decision making. A partial differential equation is derived to characterize the price of a participating policy and solved with the finite difference method. We discuss impacts of the regulation and the insurance company’s reaction to the intervention in terms of its investment strategy on the policy’s value as well as on the policyholder’s surrender behavior, which depend on the rationality level of the policyholder.

Keywords: participating life insurance policies; early default risk; surrender risk; partial rationality; regulation of financial markets

JEL classification: G22; G28

We are grateful to Nadine Gatzert, Hato Schmeiser, Judith Schneider, Michael Suchanecki, Alexander Szimayer, and Peter Zweifel for their valuable comments. This paper has also benefited from the valuable discussions by Carole Bernard, Alexander Braun, and Christoph Meinerding. We also thank An Chen and Klaus Sandmann for their encouragement and support throughout the project, as well as the participants of the joint conference of the 21st Annual Meeting of the German Finance Association and the 13th Symposium on Finance, Banking, and Insurance, the 2014 Annual Meeting of the American Risk and Insurance Association, the 41st Seminar of the European Group of Risk and Insurance Economists, the 8th Conference in Actuarial Science and Finance on Samos, and the 18th International Congress on Insurance: Mathematics and Economics for their helpful comments on earlier drafts of this paper. Financial support from the German Research Foundation through the Bonn Graduate School of Economics is gratefully acknowledged.

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1 Introduction

A typical participating life insurance policy provides policyholders with a minimum interest rate guarantee and bonus payments upon death and upon survival which are linked to the performance of the insurance company. Usually, additional options are embedded in the policies to increase their attraction to the policyholders, among which the most popular one is a surrender option. The surrender option entitles the policyholders to terminate their contracts prematurely and to obtain the surrender benefits promised by the insurance company.

The policyholders may not necessarily receive the payments specified in their contracts even if they hold the contracts until maturity. If the insurance company does not have enough reserves to pay back its liabilities at the maturity date, the policyholders cannot get more than what remains in the company. To protect the policyholders from collecting too few benefits as the insurance company declares bankruptcy at maturity, regulatory authorities impose early default mechanisms to monitor insurance companies’ financial status and close them before it is too late. For example, under Solvency II, the supervisory authority withdraws the authorization of an insurance company when its capital falls below the minimum capital requirement and does not recover within a short period of time, see Solvency II Directive (2009/138/EC). Also, an insurance company supervised by the Swiss Financial Market Supervisory Authority (FINMA) can lose its license when its risk-based capital drops below the lowest threshold specified in the Swiss Solvency Test (SST), see FINMA Circ. 08/44 SST. Proceeds from liquidated assets are then paid to stakeholders. Hence, the policyholders also face early default risk of the insurance company accompanied with the early default regulatory intervention.

Both the surrender and the early default intervention definitely have direct impacts on the fair valuation of participating life insurance policies since they change the policies’ payment streams. In the existing literature, most studies focus on only one of the two aspects. For example, Andreatta and Corradin (2003), Bacinello (2003), and Grosen and Jørgensen (2000) study the fair value of participating life insurance policies with an embedded surrender option but have not considered early default risk triggered by bad performance of the insurance company, while Bernard et al. (2005), Chen and Suchanecki (2007), Grosen and Jørgensen (2002), and Jørgensen (2001) take into account regulatory intervention in evaluating participating policies, but leave out surrender risk. The only work that treats early default risk and surrender risk at the same time is Le Courtois and Nakagawa (2013) who model the surrender risk through a Cox process of the surrender intensity which is correlated to the financial market but is independent of the company’s liquidation threshold. However, since the early termination of the insurance company imposed by the regulator reforms the contracts’ payment structure for the policyholders, which we consider as the direct impacts on the contracts’ value, as a
response the policyholders may change their surrender behaviors. Such an influence of the enforced early bankruptcy on the policyholders’ surrender behaviors can be considered as a “by-product” of the regulatory intervention, which in turn affects the contracts’ payment streams and correspondingly, the contracts’ value. In this paper, we analyze both the direct and indirect (by-product) impacts of the early default risk on the fair value of the participating policies by endogenizing the policyholders’ surrender behaviors and uncovering the impacts of the early default intervention on the surrender behaviors. Moreover, when the regulatory rule changes, the insurance company may react to the change by adopting a different investment strategy, which again affects the contracts’ value directly and indirectly through its influence on the policyholders’ surrender behaviors. Hence, we also study how the insurance company chooses its investment strategy in face of the regulatory intervention, and the impacts of the insurance company’s investment strategy on the policyholders’ surrender behaviors and their contracts’ value.

To describe the early default intervention we adopt the regulatory framework in Bernard et al. (2005), Grosen and Jørgensen (2002), and Jørgensen (2001), where liquidation is triggered as the insurance company’s asset value drops below a threshold. Concerning the surrender, in most literature, it is assumed that policyholders are fully rational, which means they can terminate their contracts at an optimal time so that the contracts’ value is maximized, see e.g., Andreatta and Corradin (2003), Bacinello (2005, 2003), and Grosen and Jørgensen (2000, 1997), to just name a few. However, since there is not an active market to monitor the contract values, the surrender option is hardly exercised at the optimal time if a policyholder is not capable of valuing the contract correctly. Also due to the lack of an active trading market for the contracts, the policyholders, when in urgent liquidity needs, have to surrender their contracts to the insurance company and collect the surrender value, which is usually lower than the contracts’ fair value. Empirical evidence which confirms the so called emergency hypothesis are found e.g. in Kiesenbauer (2011) and Kuo et al. (2003). Given these limitations, it is more reasonable to consider policyholders as partly rational from a purely financial point of view. We adopt the approach of modeling policyholders’ partial rationality in Li and Szimayer (2014). Surrender is considered as a randomized event and the arrival of the event is assumed to follow a Poisson process with the intensity bounded from below and from above. The lower and upper bounds refer to the minimum surrender rate due to exogenous reasons and the maximum surrender rate due to limited financial rationality, respectively. The maximum contracts’ value is then derived by choosing surrender intensities within the two bounds in the worst case scenario from the perspective of the insurer. This approach corresponds to the spirit of Solvency II. While valuing options written in the contracts, realistic assumptions concerning the likelihood that policyholders exercise the options should be used, see Solvency II Directive (2009/138/EC).
Moreover, CEIOPS\textsuperscript{1} has pointed out that policyholders’ surrender behaviors pose a significant risk to insurance companies and the surrender risk should be treated differently for different policyholders. For example, the surrender risk can be substantially higher if the policyholders are institutional investors since they tend to be better informed and react more likely in a financially rational way, see CEIOPS (2009). This indicates that the rationality level of the policyholders plays an important role in analyzing the fair value of their policies. Therefore, we consider different bounded values of the policyholders’ surrender intensities and analyze how the influence of the early default regulatory rule on the policyholders’ surrender behaviors differs with respect to their rationality level. Similar to Li and Szimayer (2014), we derive a partial differential equation (PDE) to characterize the price of a participating policy. However, this PDE is only valid when the liquidation threshold has not been touched yet. Otherwise, the policy takes immediately the liquidation value. In this sense, we are solving a barrier option pricing problem. We apply the finite difference method proposed in Zvan et al. (2000, 1996) to solve this problem numerically.

The paper is organized as follows. In Section 2 we model the insurance company and introduce the payoff structure of a participating policy, as well as the early default regulatory framework. Besides, both the financial market and the insurance market are modeled with respect to the stochastic processes of the underlying asset, the mortality risk intensity and the surrender risk intensity. In Section 3 we derive the PDE for the price of the policy. In Section 4 we analyze the effects of the regulatory framework and the investment strategy on the contract valuation as well as on the policyholder’s surrender behavior. Section 5 concludes.

2 Model Framework

2.1 Company Overview

Inspired by the model framework in Briys and De Varenne (1994), we consider a life insurance company which acquires an asset portfolio with initial value $A_0$ at time $t_0 = 0$ financed by two agents, i.e., a policyholder and an equity holder. The policyholder pays a premium to acquire the initial liability $L_0 = \alpha A_0$ with $\alpha \in (0, 1)$. The rest is levied from the equity holder who acquires $E_0 \equiv (1 - \alpha)A_0$ with limited liability. The insurance company’s balance sheet at time $t_0$ is shown in Table 1. The parameter $\alpha$ is called the wealth distribution coefficient in Grosen and Jørgensen (2002).

\textsuperscript{1}CEIOPS refers to the Committee of European Insurance and Occupational Pensions Supervisors. It was replaced by the European Insurance and Occupational Pensions Authority (EIOPA) since 2011.
It is assumed that the insurance company operates in a frictionless, complete and arbitrage-free financial market over a time interval \([0, T]\), where the time \(T\) corresponds to the maturity date of the insurance contract. As the insurance contract matures at \(T\), the insurance company closes and its assets are liquidated and distributed to stakeholders.\(^2\)

### 2.2 Participating Life Insurance Policy

By investing in the insurance company at time \(t_0\), the policyholder signs a participating insurance contract which promises him a share of the insurance company’s profits in addition to the guaranteed minimum interest rate at the maturity date \(T\). If the policyholder dies before time \(T\), the contract pays death benefits. Additionally, the policyholder can exercise the surrender option embedded in the contract before maturity \(T\) and collect surrender benefits from the insurance company. To summarize, the contract promises survival benefits, death benefits and surrender benefits, depending on which event happens first. In any event, the policyholder has a priority claim on the company’s assets and the equity holder receives what is left.

As the contract matures at maturity \(T\), the policyholder receives a minimum guaranteed benefit, which is given by compounding the initial liability \(L_0\) with a minimum guaranteed interest rate \(r_g\), i.e., \(L_T^{r_g} = L_0 e^{r_g T}\), and a bonus conditional on that the asset value generated by the contribution of the policyholder is enough to cover the minimum guaranteed benefit, i.e., \(\alpha A_T \geq L_T^{r_g}\). Suppose \(\delta\) is the participation rate in the asset surplus. The profits shared with the policyholder are \(\delta[\alpha A_T - L_T^{r_g}]^+\). However, it may happen that at time \(T\) when the company’s assets are liquidated, the assets’ value is lower than the value of the minimum guaranteed benefit. In this case, based on the assumptions that the policyholder has a priority claim on the company’s assets and the equity holder has limited liability, the policyholder collects what is left, i.e., \(A_T\), and the equity holder walks away with nothing in his hands. To sum up, when the contract survives until maturity \(T\), the policyholder receives survival benefits which take the form

\[
\Phi(A_T) = L_T^{r_g} + \delta[\alpha A_T - L_T^{r_g}]^+ - [L_T^{r_g} - A_T]^+. \tag{1}
\]

\(^2\)For simplicity, we assume the company closes when the contract ends. It is not a strict assumption because it can be considered that assets raised from the policyholder and the equity holder are put in a separate fund, as the contract ends, the fund is closed and assets left in the fund are liquidated and distributed to the stakeholders.
The policyholder may die before the contract matures. We use $\tau_d$ to denote the death time of the policyholder aged $x$ at time $t_0$. At time $\tau_d < T$, the contract pays death benefits to the policyholder. We assume that death benefits have the same payment structure as survival benefits, but with all the components evaluated at the death time $\tau_d$. We use $r_d$ and $\delta_d$ to denote the minimum guaranteed interest rate and the participation rate for calculating the promised minimum guarantee, i.e., $L^{\tau_d}_0 = L_0 e^{r_d \tau_d}$, and the asset surplus, respectively. Then, the death benefits have the following form at time $\tau_d$

$$\Psi(\tau_d, A_{\tau_d}) = L^{\tau_d}_0 r_d + \delta_d \left[ \alpha A_{\tau_d} - L^{\tau_d}_0 \right]^+ - \left[ L^{\tau_d}_0 - A_{\tau_d} \right]^+. \quad (2)$$

Furthermore, by exercising the surrender option embedded in the contract, the policyholder can terminate the contract before the expiration date $T$. We use $\tau_s$ to denote the surrender time. Once the surrender option is exercised, the company closes and its assets are liquidated and paid to the policyholder as specified in the contract but not more than the liquidated asset value. We consider the following surrender payment form for the policyholder

$$S(\tau_s, A_{\tau_s}) = L^{\tau_s}_0 - \left[ L^{\tau_s}_0 - A_{\tau_s} \right]^+, \quad (3)$$

where $L^{\tau_s}_0 = (1 - \beta_{\tau_s}) L_0 e^{r_s \tau_s}$ is the minimum surrender guarantee when the asset value suffices. Here, $r_s$ is the minimum guaranteed interest rate at surrender and $\beta_{\tau_s}$ is a penalty parameter which penalizes the policyholder for early terminating the contract and is assumed to be a deterministic decreasing function of the time. After the policyholder is paid off, the equity holder receives the rest of the asset value.

### 2.3 Early Default Mechanism

Now, we introduce early default risk of the insurance company into the model. We consider an external regulator who watches on the insurance company’s financial status over its operating time horizon. We abstract from cumbersome bankruptcy rules and procedures applied to insurance companies in practice and assume the insurance company is on-going until either the external regulator intervenes before $T$ or the insurance contract matures at $T$. We adopt the regulatory mechanism introduced by Grosen and Jørgensen (2002) and set up a default-triggering barrier based on the minimum survival guarantee $B_t = \theta L_0 e^{r_s t}$, where $\theta$ is a default multiplier. Once the company’s asset value drops below the barrier before maturity $T$, the company is closed by the regulator and its assets are liquidated and distributed to the stakeholders. Accordingly, we define the early default time $\tau_b$ as the first time that the asset
value drops below the barrier,

\[ \tau_b = \inf \{ t \mid A_t \leq B_t \}. \tag{4} \]

At time \( \tau_b \), the policyholder receives early default benefits, denoted by \( \Upsilon(\tau_b, A_{\tau_b}) \), which have the lower value of the liquidated assets and the minimum survival guarantee accrued at the guarantee rate \( r_g \) up to the early default time,

\[ \Upsilon(\tau_b, A_{\tau_b}) = \min\{A_{\tau_b}, L_{\tau_b}^{r_g}\}, \tag{5} \]

where \( L_{\tau_b}^{r_g} = L_0 e^{r_g \tau_b} \). Accordingly, if the company has the liquidated assets more than the promised minimum guarantee, the equity holder obtains what is left after paying off the policyholder; otherwise, the equity holder gets nothing.

The default multiplier \( \theta \) is set by the regulator, which actually reflects how intensively the regulator monitors the insurance company and how strongly the regulator intends to protect the policyholder. If the regulator believes that the insurance company is inclined to take advantage of the policyholder by running a risky business or is not competent enough to manage its assets, the regulator may set a higher default multiplier to protect the policyholder. This implies that the insurance company must bear a higher early default risk. Otherwise, the regulator will set a lower default multiplier, which allows the insurance company to recover from its temporary bad performance. In our model, we restrict \( \theta \) to be smaller than \( \frac{1}{\alpha} \), which ensures \( A_0 > B_0 \) so that the insurance company does not default at the initial time \( t_0 \) when the contract is just issued to the policyholder.

### 2.4 Mathematical Formulation

In this section we model the financial market and the insurance market mathematically. We fix a filtered probability space \( (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}) \), where \( \mathbb{F} = (\mathcal{F}_t)_{t \geq 0} \) reflects the flow of information available on the financial market and the insurance market. We assume that the company invests its total initial assets in traded (risk-free and/or risky) assets on the financial market, where the risk-free interest rate, denoted by \( r \), is assumed to be deterministic in time. Under the market probability measure \( \mathbb{P} \), the company’s asset price process \( A \) is assumed to be governed by the following stochastic process

\[ dA_t = a(t, A_t) A_t \, dt + \sigma(t, A_t) A_t \, dW_t, \quad \forall t \in [0, T]. \tag{6} \]

Here \( W \) is a standard Brownian motion under \( \mathbb{P} \) and generates the filtration \( \mathbb{F}^W = \{\mathcal{F}_t^W\}_{0 \leq t \leq T} \). The functions \( a \) and \( \sigma > 0 \) refer to the expected rate of return and the volatility of the asset process.
As the payoff of the contract depends not only on the asset value itself but also on the occurrence of the death event or the surrender event, we enlarge the filtration $\mathbb{F}^W$ in a minimal way to summarize all the information relevant to the contract valuation. The filtration $\mathbb{F}^W$ is thus enlarged to $\mathbb{G} = \mathbb{F}^W \vee \mathbb{H}$, where $\mathbb{H}$ is jointly generated by the jump processes $H_t = 1_{\{\tau_d \leq t\}}$ and $J_t = 1_{\{\tau_s \leq t\}}$, i.e., the information about whether the policyholder dies before time $t$ and whether he surrenders the contract before time $t$, respectively. The hazard rate of the random time $\tau_d$, also called mortality intensity, is denoted by $\mu$ and assumed to be a deterministic function of time.\footnote{In the literature, there are many discussions on modeling stochastic mortality intensity, see e.g., Bacinello et al. (2010), Biffis et al. (2010), Dahl (2004), Dahl and Möller (2006). However, the stochastic feature of the mortality intensity does not have too much influence on the contract value, see Li and Szimayer (2011). Therefore, in this paper, we assume a deterministic mortality intensity function for simplicity and focus more on the early default risk and the surrender risk.} Similarly, we call the hazard rate of the random time $\tau_s$ surrender intensity and denote it by $\gamma$. Taking into account that the policyholder has partial rationality in surrendering his contract, we follow the approach in Li and Szimayer (2014) by assuming the surrender intensity is bounded from below by $\bar{\rho}$, in the case that surrender is not a financially optimal decision for the policyholder, and from above by $\overline{\rho}$, in the case that surrender becomes financially optimal, with $\overline{\rho} > \rho$. Due to personal reasons which urge the policyholder to surrender the contract prematurely, the lower bound of the surrender intensity $\rho$ is present in either of the two cases, i.e., the surrender intensity at least takes the value of $\rho$. The size of the increase in the surrender intensity as surrendering becomes financially optimal to the policyholder, i.e., $\bar{\rho} - \rho$, measures how frequently the policyholder updates his financial market information and how likely he makes the surrender decision when it is optimal to do so, which is also called endogenous surrender intensity. The more frequently he updates and analyzes financial information, the larger the increase in the intensity is, accordingly, the more rational he is to make his surrender decision. In the case of $\bar{\rho} = \infty$, the policyholder surrenders immediately when it is optimal to do so and together with a zero exogenous surrender rate $\rho = 0$, we are back to the case of pricing an American-style contract by solving an optimal stopping problem. The decision is made by comparing the continuation value of the contract and the value of surrender benefits, which are denoted by $v(t, A)$ and $S(t, A)$, respectively. Depending on which decision the policyholder makes, the endogenous surrender intensity is thus either $0$ or $\bar{\rho} - \rho$. To summarize, the surrender intensity takes the form of

$$
\gamma_t = \begin{cases} 
\rho, & \text{for } S(t, A) < v(t, A) \\
\bar{\rho}, & \text{for } S(t, A) \geq v(t, A).
\end{cases}
$$

(7)

Conditional on the current information available on the financial market and the insurance process respectively, and both are regular enough to guarantee a unique solution of (6).
market, the arrival of the death event, the arrival of the surrender event, and $W$ are independent.
Thus, $W$ is a $\mathbb{G}$-martingale, and $\mu$ and $\gamma$ are $\mathbb{G}$-intensities of the random death time $\tau_d$ and the random surrender time $\tau_s$, respectively.

In the absence of arbitrage, we use the risk-neutral pricing approach with a martingale measure $\mathbb{Q}$ to price the participating life insurance contract. Under the martingale measure $\mathbb{Q}$, the company’s asset process is described by

$$dA_t = r(t)A_t dt + \sigma(t,A_t)A_t dW^Q_t, \quad \forall t \in [0,T],$$

(8)

where $W^Q$ is a standard Brownian motion. Taking the mortality risk and the surrender risk, with $\mu$ and $\gamma$ as $\mathbb{G}$-intensities of the random times $\tau_d$ and $\tau_s$, respectively, into consideration, pricing the participating life insurance contract under the martingale measure $\mathbb{Q}$ requires additional justification. Under the condition that the mortality intensity is deterministic, if the pool of policyholders is large enough, mortality risk is diversifiable for the insurer, and thus $\mu$ is the $(\mathbb{Q},\mathbb{G})$-intensity of mortality. The surrender intensity specified in (7) corresponds to the worst-case scenario from the insurance company’s perspective. As long as we assume the insurance company does not ask for an extra risk premium above the worst-case surrender intensity under the martingale measure $\mathbb{Q}$, the bounds $\rho$ and $\bar{\rho}$ are still valid under the measure $\mathbb{Q}$. Then the surrender intensity specified in (7) corresponds to the $(\mathbb{Q},\mathbb{G})$-intensity of surrender on the enlarged market represented by the filtration $\mathbb{G}$. The contract value obtained under the measure $\mathbb{Q}$ with the worst-case surrender intensity $\gamma$ can be interpreted as the upper price bound of the contract. We address this issue formally in Remark 1. Consequently, the choice of the surrender intensity in (7) is not only motivated by the observations of policyholders’ surrender behaviors on the market but also by the worst-case scenario analysis within a reasonable range that is often adopted in practice, see CEIOPS (2009).

3 Contract Valuation

In this section we value the contract by taking both the early default risk and the surrender risk into consideration. Since policyholder’s surrender behavior is endogenously modeled in this paper, the surrender intensity can only be determined endogenously. By applying the PDE approach, we can specify the surrender intensity and the contract value at the same time. Moreover, after introducing the early default mechanism, the contract payoff to the policyholder

4Alternatively, a higher market price for the surrender risk may be charged by lowering the lower bound $\rho$ and increasing the upper bound $\bar{\rho}$ under the measure $\mathbb{Q}$. In Proposition 2 we show formally that a lower $\rho$ and a higher $\bar{\rho}$ lead to a higher contract value.

5Confer Li and Szimayer (2014) for a formal explanation of the surrender intensity after the change of measure.
is connected to the solvency of the company and has a barrier option feature. Thus, we need to
distinguish the case where the insurance company is ongoing and the case where the regulator
intervenes, which means that, in order to value the contract, we differentiate between the region
where \( A_t \leq B_t \) and the region where \( A_t > B_t \) for \( t \in (0, T) \). This is similar to the barrier
option pricing. For \( A_t \leq B_t \) at time \( t \in (0, T) \), the insurance company must be liquidated and
the policyholder only obtains \( \Upsilon(t, A_t) \). For \( A_t > B_t \), we represent the contract value \( V_t \)
on \( \{ t \leq \tau_d \wedge \tau_s \wedge T \} \) by

\[
V_t = \mathbb{1}_{\{t < \tau_d \wedge \tau_s \wedge \tau_b \wedge T \}} v(t, A_t) + \mathbb{1}_{\{t = \tau_d, \tau_d < \tau_s \wedge T \}} \Psi(t, A_{\tau_d}) + \mathbb{1}_{\{t = \tau_s, \tau_s < \tau_d \wedge T \}} S(\tau_s, A_{\tau_s}),
\]

where \( v \) is a suitably differentiable function \( v : [0, T] \times \mathbb{R}^+ \rightarrow \mathbb{R}^+ \), \( (t, A) \mapsto v(t, A) \),
representing the pre-death/surrender value. Then we apply the no-arbitrage pricing condition
on the set \( \{ t < \tau_d \wedge \tau_s \wedge \tau_b \wedge T \} \), being

\[
r(t) V(t, A_t) dt = \mathbb{E}_Q[dV_t | \mathcal{G}_t].
\]

On the set \( \{ t < \tau_d \wedge \tau_s \wedge \tau_b \wedge T \} \), we compute the differential of \( V \) as\(^{6}\)

\[
dV_t = dv(t, A_t) + (\Psi(t, A_t) - v(t, A_t))dH_t + (S(t, A_t) - v(t, A_t))dJ_t,
\]

where \( H \) and \( J \) refer to the jump processes with the \( Q \)-intensities \( \mu \) and \( \gamma \), respectively. A jump
in \( H \) or \( J \) leads to a change in the payment liability either of the amount \( \Psi(t, A_t) - v(t, A_t) \) or
\( S(t, A_t) - v(t, A_t) \). Plugging (11) into (10) and using \( V_t = v(t, A_t) \) at time \( t < \tau_d \wedge \tau_s \wedge \tau_b \wedge T \),
we obtain

\[
r(t) v(t, A_t) dt = \mathbb{E}_Q[dv(t, A_t) | \mathcal{G}_t] + (\Psi(t, A_t) - v(t, A_t))\mu(t)dt + (S(t, A_t) - v(t, A_t))\gamma(t)dt.
\]

By applying Ito’s Lemma to \( dv(t, A_t) \), we have

\[
\mathbb{E}_Q[dv(t, A_t) | \mathcal{G}_t] = \mathbb{E}_Q \left[ \mathcal{L}v(t, A_t)dt + \sigma(t, A_t)A_t \frac{\partial v}{\partial A}(t, A_t)dW^Q_t | \mathcal{G}_t \right]
\]

\[
= \mathcal{L}v(t, A_t)dt,
\]

\(^{6}\)Notice that in the region \( A_t > B_t \), there would not be early default after the instantaneous time period \( dt \)
since the asset process is assumed to be continuous in our model.
where \( \mathcal{L}v(t, A) = \frac{\partial v}{\partial t}(t, A) + r(t)A \frac{\partial v}{\partial A}(t, A) + \frac{1}{2}\sigma^2(t, A)A^2 \frac{\partial^2 v}{\partial A^2}(t, A) \). Then, on the set \( \{t < \tau_d \land \tau_s \land \tau_b \land T\} \) we have

\[
\mathcal{L}v(t, A_t) + \mu(t)\Psi(t, A_t) + \gamma(t)S(t, A_t) - (r(t) + \mu(t) + \gamma(t))v(t, A_t) = 0. \tag{14}
\]

We summarize the pricing PDE in the following proposition.

**Proposition 1.** For the contract value \( V \) described by (9), the pre-death/surrender value \( v \) for \( (t, A_t) \in [0, T) \times \mathbb{R}^+ \) is the solution of the partial differential equation

\[
\mathcal{L}v(t, A_t) + \mu(t)\Psi(t, A_t) + \gamma(t)S(t, A_t) - (r(t) + \mu(t) + \gamma(t))v(t, A_t) = 0, \tag{15}
\]

where

\[
\gamma(t) = \begin{cases} \rho, & \text{for } S(t, A_t) < v(t, A_t), \\ \bar{\rho}, & \text{for } S(t, A_t) \geq v(t, A_t); \end{cases} \tag{16}
\]

subject to the boundary condition

\[
v(t, A_t) = \Upsilon(t, A_t), \text{ for } t \in [0, T), \ A_t = B_t = \theta L_0 e^{r s t}, \tag{17}
\]

and the termination condition

\[
v(T, A_T) = \Phi(A_T), \text{ for } A_T \in \mathbb{R}^+. \tag{18}
\]

The integral representation of the solution to the above pricing PDE is shown in Corollary 1 and proved in Appendix A.

**Corollary 1.** Suppose the surrender intensity \( \gamma \) is given. The value of the participating policy \( V \) can be represented on \( \{t < \tau_s \land \tau_d \land \tau_b \land T\} \) by

\[
V_t = \mathbb{E}_Q \left[ \int_t^{\tau_b \land T} e^{-\int_t^m (r(u) + \mu(u) + \gamma(u, A_u)) du} (\mu(m)\Psi(m, A_m) + \gamma(m, A_m)S(m, A_m)) dm \\
+ 1_{\{\tau_b \geq T\}} \Phi(A_T) e^{-\int_t^T (r(u) + \mu(u) + \gamma(u, A_u)) du} + 1_{\{\tau_b < T\}} \Upsilon(\tau_b, A_{\tau_b}) e^{-\int_t^{\tau_b} (r(u) + \mu(u) + \gamma(u, A_u)) du} \right] \left| G_t \right]. \tag{19}
\]

**Remark 1.** The pricing problem can be formulated as looking for the worst case of the risk-adjusted surrender intensity \( \gamma \) so that the contract value is maximized under the martingale
measure $Q$ on \( \{ t < \tau_s \wedge \tau_d \wedge \tau_b \wedge T \} \),

\[
v(t, A) = \sup_{\gamma \in \Gamma(t, A)} E_t^t_A \left[ \int_t^{\tau_b \wedge T} e^{-\int_u^t (r(u) + \mu(u) + \gamma(u, A_u))du} (\mu(m)\Psi(m, A_m) + \gamma(m, A_m)S(m, A_m))dm \\
+ 1_{\{ \tau_b \geq T \}} \Phi(A_T)e^{-\int_T^t (r(u) + \mu(u) + \gamma(u, A_u))du} + 1_{\{ \tau_b < T \}} \Upsilon(\tau_b, A_{\tau_b})e^{-\int_{\tau_b}^t (r(u) + \mu(u) + \gamma(u, A_u))du} \right],
\]

where \( \Gamma(t, A) = \{ \gamma : [t, T] \times \mathbb{R}^+ \rightarrow \mathbb{R}^+ : \rho \leq \gamma(u, A) \leq \bar{\rho}, \text{ for all } t \leq u \leq T \text{ and } A \in \mathbb{R}^+ \} \) and \( E_t^t_A \) denotes the expectation conditional on \( A_t = A \) under the measure \( Q \). This is a stochastic control problem, which can be solved, according to the theorem of the Hamilton-Jacobi-Bellman equation (confer Yong (1997) and Yong and Zhou (1999)), by dealing with an equivalent problem

\[
0 = \sup_{\gamma \in \Gamma(t, A)} \mathcal{L}v(t, A) + \mu(t)\Psi(t, A) + \gamma(t, A)S(t, A) - (r(t) + \mu(t) + \gamma(t, A))v(t, A),
\]

subject to \( v(t, A) = \Upsilon(t, A) \), for \( A = B_t = \theta L_0 e^{\sigma t} \), and \( v(T, A) = \Phi(A) \), for \( A \in \mathbb{R}^+ \). \( \gamma \) needs to be optimally controlled: Since in the equation above the part that depends on \( \gamma \) is linear in \( \gamma \), i.e., \( \gamma(t, A)(S(t, A) - v(t, A)) \), the solution to the problem is exactly the same as is presented in equation (7).

Within a given regulatory framework and under a given investment strategy, i.e., for given \( \theta \) and \( \sigma \), we prove that a lower value of \( \rho \) or a higher value of \( \bar{\rho} \) leads to an increase of the contract value, see Proposition 2. This is consistent with our intuition, since a lower \( \rho \) or a higher \( \bar{\rho} \) indicates the increase of the rationality level of the policyholder in the monetary sense and thus increases the contract value. The proof is provided in Appendix B.

**Proposition 2.** Suppose the early default mechanism is characterized by the default multiplier \( \theta \) and the insurance company’s investment strategy by \( \sigma \). Furthermore, suppose that \( v \) is the pre-death/surrender value function of the participating policy with bounds of the surrender intensity being \( \underline{\rho} \) and \( \bar{\rho} \), and that \( w \) is the pre-death/surrender value function of the policy with bounds \( \underline{\zeta} \) and \( \bar{\zeta} \). Assume that \( \underline{\zeta} \leq \underline{\rho} \) and \( \bar{\rho} \leq \bar{\zeta} \). Then we have \( w(t, A) \geq v(t, A) \), for \( (t, A) \in [0, \tau_b \wedge T] \times \mathbb{R}^+ \).

### 4 Numerical Analysis

In this section, we adopt the finite difference method proposed by Zvan et al. (2000, 1996) to numerically solve the PDE with a continuously applied barrier (15) as stated in Proposition 1
and study the impacts of the early default risk and the surrender risk on the fair valuation of the contract as well as on the insurance company’s investment strategy. The insurance company is set up with initial asset value $A_0 = 100$ and 85% of the asset value is acquired by the policyholder who buys the participating contract at time $t_0$ as the initial liability, which means $\alpha = 0.85$. The contract matures in $T = 10$ years and promises the same participation rate $\delta = \delta_d = 0.9$ at maturity and at death.\footnote{Regulators usually require the participation rate to be kept at least at a certain level. In Germany, e.g., it lies at 90%.} The risk-free interest rate is $r = 0.04$ and the volatility of the company’s asset process is constant, i.e., $\sigma(t, A_t) = 0.2$.\footnote{The values of the risk-free interest rate and the volatility are chosen purely for the illustration purpose.} The volatility provides information about the riskiness of the insurance company’s investment strategy. A higher $\sigma$ indicates a higher riskiness of the investment strategy while a lower $\sigma$ implies a more conservative investment strategy.\footnote{On a financial market with only one risk-free and one risky asset, a higher asset volatility is achieved by investing more into the risky asset.} The minimum guaranteed interest rates at survival, at death and at surrender are $r_g = r_d = r_s = 0.02$. As for the mortality intensity, we follow the Gompertz-Makeham law by assuming a deterministic process $\mu(t) = A\mu + Bc^x + t$ for the policyholder aged $x = 40$ at $t_0 = 0$ with $A\mu = 5.0758 \times 10^{-4}$, $B = 3.9342 \times 10^{-5}$, $c = 1.1029^{10}$. Additionally, the penalty parameter takes the form

$$\beta_t = \begin{cases} 
0.05, & \text{for } t \leq 1, \\
0.04, & \text{for } 1 < t \leq 2, \\
0.02, & \text{for } 2 < t \leq 3, \\
0.01, & \text{for } 3 < t \leq 4, \\
0, & \text{for } t > 4. 
\end{cases}$$

The parameters are summarized in Table 2.

<table>
<thead>
<tr>
<th>Market Parameters</th>
<th>Contract Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_0$</td>
<td>$\alpha$</td>
</tr>
<tr>
<td>100</td>
<td>0.85</td>
</tr>
<tr>
<td>$r$</td>
<td>$T$</td>
</tr>
<tr>
<td>0.04</td>
<td>10</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>$\delta, \delta_d$</td>
</tr>
<tr>
<td>0.2</td>
<td>0.9</td>
</tr>
<tr>
<td>$A\mu$</td>
<td>$r_g, r_d, r_s$</td>
</tr>
<tr>
<td>$5.0758 \times 10^{-4}$</td>
<td>0.02</td>
</tr>
<tr>
<td>$B$</td>
<td>$c$</td>
</tr>
<tr>
<td>$3.9342 \times 10^{-5}$</td>
<td>1.1029</td>
</tr>
</tbody>
</table>

<p>| | |</p>
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<tbody>
<tr>
<td>Table 2: Parameter specifications</td>
<td></td>
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</table>

The analysis in the following subsections is conducted for a representative policyholder.

\footnote{Source of the parameter values: Delbean (1986).}
Under the assumption that the pool of policyholders is large enough, the surrender intensity of a representative policyholder gives an indication of the proportion of policyholders who will surrender their contracts at the portfolio level. The implications for a large pool of policyholders will be summarized in Section 5 to conclude the paper.

4.1 Effects of Regulatory Framework on Contract Valuation

In this section, we analyze the effects of the early default risk on the fair valuation of the contract. The magnitude of the early default risk depends on the strictness of the regulatory framework, which in our model is represented by the default multiplier $\theta$ that is specified by the regulator. It indicates how the regulator judges the insurance company’s ability to manage its assets. If the regulator has confidence in the expertise of the insurance company and about the financial market, she will tolerate a temporary poor performance of the insurance company and hence choose a lower default multiplier giving the company has the chance to recover. Otherwise, she will set a higher value to protect the policyholder from not being able to obtain the guaranteed benefits promised by the company. Although a lower (higher) default multiplier is less (more) effective to protect the policyholder from a downside development of the company, it gives the company a higher (lower) chance to recover from the temporary bad performance and pay out more benefits (less benefits) to the policyholder when it recovers. Hence, the level of the default multiplier has great influence on the payoff of the contract and thus on the contract value.

Furthermore, the policyholder takes into account the impacts of the protection from the regulator on his contract’s payments (eventually, on his contract value) and adjusts his surrender behavior accordingly, which indirectly influences his contract value. Intuitively, the policyholder makes his surrender decision based not only on benefits that are promised by the insurance company but also on the ability of the insurance company to meet its promise. The early default mechanism ensures the ability of the insurance company to meet its promise by imposing a limit on its asset value. A higher default multiplier implies that the policyholder has to worry less about the second issue because he is better protected and will surrender the contract only when the surrender benefits are very attractive to him. On the contrary, if the default multiplier is set lower so that the policyholder would not be protected completely, he must take the default risk of the insurance company seriously into account when implementing his surrender strategy. In this case, the policyholder may be willing to surrender his contract earlier to avoid losing too much of his initial investment.

Table 3 presents the contract value for different values of the default multiplier $\theta$ and different rationality levels represented by $(\rho, \bar{\rho})$. In the second column are the contract values in the case when there is no early default mechanism. From the third to the fifth column are the
contract values with different levels of regulatory strength which are represented by the different values of the default multiplier $\theta$. For example, $\theta = 0.7$ means that the regulator does not allow the insurance company’s asset value to drop below 70% of the minimum guarantee. $\theta = 1.1$ indicates that the regulator is more conservative and requires the company’s asset value to lie above 110% of the minimum guarantee. Comparing the contract values in columns 2-4 where the early default regulation is introduced, and then strengthened, the contract value increases gradually for all types of policyholders. Introducing the early termination rule protects the policyholder from the downside risk of the insurance company’s investment and increasing the default multiplier enlarges the protection level. An interesting feature is that the effects of the early termination regulation not only depend on the default multiplier $\theta$ but also on the rationality level of the policyholder. For example, a policyholder with $(\rho, \bar{\rho}) = (0, 0)$ never surrenders his contract, which turns to be a European-type contract. In this case that the policyholder would be better off if he terminates his contract and collects the surrender benefits when his contract value drops, imposing an early default regulatory rule such as $\theta = 0.7$ helps the policyholder improve his position. However, the benefits from the regulator’s protection become smaller as the policyholder becomes financially more rational. The default multiplier does not play a significant role when the policyholder is able to exercise the surrender option optimally, i.e. $(\rho, \bar{\rho}) = (0, \infty)$.\textsuperscript{11} Since the fully rational policyholder can find the optimal surrender strategy anyway, he does not need the protection of the regulator. However, the positive effect of strengthening the early default regulation disappears as an over-regulation rule is carried out. As the default multiplier increases from 0.9 to 1.1, the contract value decreases in some cases, e.g., when $(\rho, \bar{\rho}) = (0, \cdot), (\rho, \bar{\rho}) = (0.03, \infty)$, among which we can even observe the disadvantage of introducing the early default regulation. For the policyholder

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
 & no early default & with early default \tabularnewline & $\theta = 0.7$ & $\theta = 0.9$ & $\theta = 1.1$ \tabularnewline \hline
(0, 0) & 85.6141 & 86.8199 & 90.4937 & 89.6619 \tabularnewline (0, 0.03) & 86.0368 & 87.0668 & 90.5088 & 89.6619 \tabularnewline (0, 0.3) & 88.1531 & 88.4680 & 90.6270 & 89.6619 \tabularnewline (0, $\infty$) & 92.0546 & 92.0548 & 92.0628 & 89.6619 \tabularnewline (0.03, 0.03) & 81.8567 & 82.9119 & 86.6744 & 87.9197 \tabularnewline (0.03, 0.3) & 84.2656 & 84.5696 & 86.8343 & 87.9197 \tabularnewline (0.03, $\infty$) & 88.5391 & 88.5392 & 88.5436 & 87.9197 \tabularnewline (0.3, 0.3) & 75.4561 & 75.7496 & 78.0482 & 83.2947 \tabularnewline (0.3, $\infty$) & 80.7500 & 80.7500 & 80.7500 & 83.2951 \tabularnewline \hline
\end{tabular}
\caption{Contract values for different default multipliers $\theta$ and different rationality levels represented by $(\rho, \bar{\rho})$}
\end{table}

\textsuperscript{11}The contract values are all around 92 for $(\rho, \bar{\rho}) = (0, \infty)$ and $\theta = \{0, 0.7, 0.9\}$. 

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with \((\rho, \bar{\rho}) = (0, \infty)\) and \((\bar{\rho}, \bar{\rho}) = (0.03, \infty)\), the contract value becomes even lower than when there is no early default risk. As we have mentioned in Section 2.4 that the policyholder with \(\bar{\rho} = \infty\) may surrender the contract at any time when it is optimal to do so, irrespective of exogenous reasons, he is able to protect himself from the downside risk of the company’s investment. However, enforcing a termination regulation with a very large default multiplier stops him from obtaining more benefits in the favorable development of the insurance company, which actually lowers the contract value. Additionally, if we take a look at the column of the contract values for \(\rho = 0\) and \(\theta = 1.1\), the contracts have the same value. Since when the default multiplier is so high that the benefits obtained by the policyholder at liquidation are higher than the surrender benefits, surrendering the contract becomes unattractive, which means there would also be no endogenous reason for the policyholder to surrender his contract prematurely. Therefore, if the policyholder does not surrender his contract for exogenous reasons \((\rho = 0)\), the contract value stays at the same level as the European-style contract value 89.6619, no matter how financially rational the policyholder is.

We have discussed in Section 1 that due to personal reasons, the policyholder surrenders his contract even though he is capable of valuing his contract and understands surrender results in a financial loss. Such exogenous surrender lowers the contract’s fair value. We isolate the impact of exogenous surrender on the contract’s fair value by calculating the decrease in the contract value as \(\rho\) deviates from 0 when \(\bar{\rho} = \infty\). The decrease in the contract value actually measures the premium that the insurance company should not have charged the policyholder due to his personal non-avoidable liquidity reasons. We call this premium the liquidity premium. In Figure 1 we present the liquidity premium for different values of the regulatory default multiplier and the exogenous surrender intensity. We observe the following trends. First, within the same regulatory framework, the liquidity premium becomes larger as the exogenous surrender intensity increases. It implies that as the exogenous surrender intensity increases, the insurance company needs to compensate the policyholder more in terms of lowering the contract value in order to make the contract more attractive to him. Second, the liquidity premium increases faster at a lower \(\theta\)-level while more slowly at a higher \(\theta\)-level, until the exogenous surrender intensity \(\rho\) also becomes quite large. This indicates that the value of the liquidity premium is more sensitive to the policyholder’s exogenous surrender intensity level \(\rho\) at a lower \(\theta\)-level, where the protection from the regulator is low and it is more necessary for the regulator to urge the insurance company to assess policyholder’s exogenous surrender rate more precisely. On the contrary, as the intervention by the regulator is enhanced, the probability that the insurance company is closed increases. Liquidation may happen before the policyholder exercises the surrender option due to the exogenous reasons. Since the policyholder is not penalized at liquidation, he may receive more than the surrender guarantee he may otherwise obtain from
surrendering his contract.

Similar to the above discussion on the impact of the exogenous surrender intensity on the contract value, the endogenous surrender intensity also influences the contract value. Since the policyholder has limited information on the financial market and limited knowledge to value the contract on his own, i.e., \( \bar{\rho} < \infty \), he may fail to surrender the contract when he should do so, which results in a decrease in his contract value. Similarly, we isolate the impact of endogenous surrender on by calculating the decrease in the contract value as the upper bounder surrender intensity \( \bar{\rho} \) deviates from infinity while setting \( \rho = 0 \). This decrease in the contract value measures the premium that the insurance company should not have charged the policyholder due to his limited information and limited valuation ability, which we name rationality premium. In Figure 2 we plot the rationality premium as a function of the upper bound surrender intensity \( \bar{\rho} \) and the default multiplier \( \theta \). It is natural to observe that the rationality premium decreases in \( \bar{\rho} \) at a given protection level \( \theta \) determined by the regulator, which is consistent with Proposition 2. Furthermore, for a given value of \( \bar{\rho} \), the rationality premium decreases as the regulator’s intervention level is enhanced until the default multiplier \( \theta \) reaches 1. Intuitively, the intervention by the regulator helps the policyholder close his contract prematurely by shutting down the insurance company as its asset value drops below the liquidation threshold, which is often the timing that the policyholder should have surrendered his contract but failed to do so. In addition, such an intervention by the regulator does not bring penalties to the policyholder, which further improves his financial position. Therefore, with the
protection from the regulator and as the protection enlarges, the insurance company charges the policyholder a higher contract value, which implies a lower a lower value of the rationality premium. But in a non-strict regulation environment, i.e., where the early default threshold is low, it is important for the insurance company to assess the endogenous surrender intensity of the policyholder so that the contract price is reduced enough to attract the policyholder.

Figure 2: Rationality premium as a function of the upper bound surrender intensity $\bar{\rho} \in [0.3, 30]$ and the default multiplier $\theta \in [0, 1.1]$

### 4.2 Effects of Insurance Company’s Investment Strategy on Contract Valuation

In this section we analyze the effects of insurance company’s investment strategy on the contract valuation and discuss insurance company’s risk-shifting investment strategy in two environments, namely with and without the early default intervention by the regulator. The investment strategy is represented by the volatility $\sigma$ of the underlying asset $A$. The higher the volatility $\sigma$, the higher the risk that the insurance company has entered into. Table 4 presents the contract value for different values of $\sigma$ and different rationality levels represented by $(\rho, \bar{\rho})$ with and without the early default intervention. The early default multiplier $\theta$ is 0.9.\(^\text{12}\)

The effects of the company’s investment strategy on the contract valuation appear to be different within two regulatory frameworks. For the no early default case, we observe three tendencies, which depend on the policyholder’s rationality level. When $(\rho, \bar{\rho}) = (0, \infty)$ or

\(^\text{12}\)We have also studied the cases with $\theta = 0.7$ and 1.1. However, we have not found any qualitative differences in the effects of the volatility $\sigma$ and hence do not present all the results here.
(0.03, ∞), the policyholder is considered to be financially rational because the policyholder will exercise the surrender option once his contract value is higher than the surrender value. An increase in the underlying asset risk also implies a potentially higher expected rate of return, which can be easily captured by a rational policyholder. Hence, the contract value increases in the underlying asset risk. For (ρ, \bar{\rho}) = (0, 0), (0, 0.03), (0, 0.3) or (0.03, 0.3), which indicates either zero or very low exogenous surrender intensity value, and bounded endogenous surrender intensity, we observe an increase first and then a decrease in the contract value as σ increases.

Intuitively, when the underlying asset risk increases but still stays at a lower level, the downside risk is still limited and the optimal surrender intensity during the contract’s life time stays at a lower level anyway. However, the chance to participate in the favorable development of the asset value increases. Hence, overall the contract value increases slightly when σ increases from 0.1 to 0.2. However, as the asset risk increases further, the downside risk could be so high that it is necessary to check more frequently whether to surrender the contract or not. The bounded endogenous surrender intensity in this case would then lead to a lower contract value for the policyholder. When (\bar{\rho}, \rho) = (0.03, 0.03) or (0.3, 0.3), the endogenous surrender intensity is zero and the policyholder surrenders his contract only for exogenous reasons, which are not related to the contract value at all. Higher investment risk requires a more rapid and correct response to changing market conditions. When the policyholder is not willing to do so or capable of doing so, the contract value for the policyholder will decrease in the investment risk, represented by σ.

Now as the early default mechanism is implemented by the regulator, the contract value increases as the volatility σ increases in most cases, except when the probability that the policyholder surrenders his contract due to exogenous reasons is relatively large, i.e., \bar{\rho} = 0.3. In Table 4, we observe that the contract value decreases in σ for (\bar{\rho}, \rho) = (0.3, 0.3), and stays

<table>
<thead>
<tr>
<th></th>
<th>no early default</th>
<th>with early default</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>σ = 0.1</td>
<td>σ = 0.2</td>
</tr>
<tr>
<td>(0, 0)</td>
<td>85.3380</td>
<td>85.6141</td>
</tr>
<tr>
<td>(0, 0.03)</td>
<td>85.5737</td>
<td>86.0368</td>
</tr>
<tr>
<td>(0, 0.3)</td>
<td>86.7156</td>
<td>88.1531</td>
</tr>
<tr>
<td>(0, ∞)</td>
<td>88.3422</td>
<td>92.0546</td>
</tr>
<tr>
<td>(0.03, 0.03)</td>
<td>82.8209</td>
<td>81.8567</td>
</tr>
<tr>
<td>(0.03, 0.3)</td>
<td>84.0278</td>
<td>84.2656</td>
</tr>
<tr>
<td>(0.3, 0.3)</td>
<td>78.2582</td>
<td>75.4561</td>
</tr>
<tr>
<td>(0.3, ∞)</td>
<td>80.7500</td>
<td>80.7500</td>
</tr>
</tbody>
</table>

Table 4: Contract values for different investment strategies represented by σ and different rationality levels represented by (\rho, \bar{\rho}), \theta = 0.9.
constant for \((\rho, \bar{\rho}) = (0.3, \infty)\). In the two cases, it happens that the policyholder surrenders his contract even when the asset value increases, which deprives him of the chance to participate in the asset appreciation. Since the policyholder is protected by the regulator through the early default barrier, the potential downside risk of the insurance company’s investment is limited while the potential participation in the favorable asset performance is still possible. As long as the policyholder is not rushing to liquidate his contract, he can benefit more from the regulator’s protection as the riskiness of the investment strategy increases and his contract value increases accordingly.

Similar to Section 4.1, we present in Figure 3 the liquidity premium for different investment strategies (adopted by the insurance company) both in the case of no early default intervention by the regulator, see Figure 3 (a), and the case where there is an early default mechanism, see Figure 3 (b). We see that, for a given rationality level \((\rho, \bar{\rho})\), liquidity premium increases in the volatility of the underlying asset in both cases. As the investment risk of the insurance company increases, the probability that the policyholder sells his contract due to exogenous reasons back to the insurance company which has been experiencing financial difficulties becomes higher. This indicates that increasing the riskiness of the investment generally does harm to the policyholder who is likely to cash out his contract due to personal non-avoidable reasons. Hence, the insurance company lowers the contract price to attract the policyholder as it increases the riskiness of its investment. It happens in both cases with and without the early default regulation. However, with the protection of the regulator, the downside risk of the insurance company’s investment is limited. Therefore, the increase in the contract value as the volatility \(\sigma\) increases is smaller than in the case of no protection of the regulator.

We present the rationality premium depending on the investment strategy and the endogenous surrender intensity level in Figure 4. We see that, given an endogeneous surrender intensity value \(\bar{\rho}\), the rationality premium increases monotonically with the riskiness of the investment strategy \(\sigma\) when there is no early default intervention by the regulator. The rationality premium is much higher when the endogenous surrender intensity level \(\bar{\rho}\) is low. Unlike a fully rational policyholder who can track the financial performance of the insurance company and act optimally to maximize his benefits, a partially rational policyholder faces the risk of mistakenly holding a contract whose value is lower than the surrender value. Such risk increases as the company’s asset process becomes more volatile and is reflected by the increasing rationality premium with respect to \(\sigma\). However, when the early default mechanism is imposed, the rationality premium first increases and then decreases in \(\sigma\). The decreasing effect can be explained by the regulator’s intervention or protection, which as a remedy for the policyholder’s “insufficient” surrender intensity, induces the insurance company to charge the policyholder a higher contract price and lower the rationality premium accordingly.
Due to the influence of the policyholder’s rationality level and the regulatory framework on the contract value, as a response, the insurance company may change its investment strategy. We assume that the insurance company performs in the interest of the equity holder. Since the contract value can be regarded as the market value of the insurance company’s liabilities when the insurance company is ongoing, the objective of the company, maximizing the residual value for the equity holder, is thus to minimize the value of the policyholder’s policy. From Table 4 we can infer which investment strategy the insurance company tends to adopt. If there is no early default regulatory rule and the rationality level of the policyholder is very high,
the insurance company prefers to take a low-risk investment. This gives us two implications. First, if the policyholder is rational enough to surrender his contract, the regulator, aiming at inducing the insurance company to avoid too risky investment, does not need to interfere with the insurance company’s investment decision anymore. Second, looking back into history, insurance companies have not always taken conservative investment strategies. An aspect that we can infer from our study is that the insurance company has actually assumed that the policyholder will not always act optimally. Considering this, it is then inappropriate to price the surrender option as a pure American-style option as it is often assumed in the literature, since the policy tends to be overpriced under this assumption which is unfair for the policyholder. Hence, if the insurance company chooses parameters for pricing the contract such that the contract value is exactly equal to the policyholder’s payment by assuming a high rationality level and leading us to think that it will adopt an investment strategy with low risk under its pricing assumption, the company actually has the incentive to increase the riskiness of its investment strategy afterwards. This problem will be avoided most likely as the early default regulation is introduced. We can read out from Table 4 that the insurance company prefers a low-risk investment in all cases but one when the early default regulation is present.

4.3 Effects of Regulatory Frameworks and Investment Strategies on Surrender Behaviors

To demonstrate the effects of the regulatory framework on the policyholder’s surrender behavior, we depict in Figure 5 the separating boundaries which illustrate the regions where the policyholder surrenders the contract for exogenous reasons and the regions where the policyholder surrenders the contract for endogenous reasons. When the early default regulation is enforced, part of the surrender region will be replaced by the early default region.

Based on our consideration about the policyholder’s rationality level as is illustrated in Section 4.1 and 4.2, we assume $\rho = 0.03$ and $\bar{\rho} = 0.3$. We begin with the graph for the case where there is no early default regulatory rule, see Figure 5 (a). Here we can observe three regions. When the asset price $A$ is relatively high, the policyholder only surrenders for exogenous reasons, because participation in insurance company’s favorable asset performance is very attractive, which is, according to the contract design, only possible when the policyholder holds the contract until death or until maturity. When asset price $A$ is very low, there would also only be exogenous surrender. This is because in this case participation in company’s favorable asset performance at death or at maturity is hardly possible and early termination of the policy carries penalty on the minimum guarantee, the policyholder would rather hold his contract if he does not have other exogenous surrender reasons. The region in the middle of the graph
Figure 5: Separating boundaries of the policyholder’s surrender behavior both in the case of no early default intervention and in the case of the early default intervention, $\theta = \{0.7, 0.9, 1.1\}$
corresponds to the case when the policyholder surrenders his contract for endogenous reasons. In this region the probability that the policyholder surrenders the contract increases mainly due to the reason that the policyholder wants to protect himself from the potential downside risk when it is still not too late to do so. If the regulator intervenes, see Figure 5 (b)-(d), we observe that the region with $\tilde{\rho} = 0.3$ is more and more replaced by the early default which is triggered by the regulatory rule. When $\theta = 1.1$ the policyholder only surrenders the contract for exogenous reasons. This is consistent with the results presented in Figure 2, where the rationality premium reduces to 0 when $\theta = 1.1$.

![Figure 6: Separating boundaries of the policyholder’s surrender behavior for different investment strategies represented by $\sigma$ when there is no early default intervention](image)

(a) $\sigma = 0.1$
(b) $\sigma = 0.2$
(c) $\sigma = 0.3$

In Figure 6 and Figure 7 we present the separating boundaries of the policyholder’s surrender behavior as the volatility increases from 0.1 to 0.3 when there is no early default risk and when there is early default risk, respectively, assuming the rationality level of the representative policyholder is $(\rho, \tilde{\rho}) = (0.03, 0.3)$. The first interesting observation is that the region $\tilde{\rho} = 0.3$ is larger in the case where there is no early default risk than in the case where there is early default risk for every asset risk level $\sigma$. Since the policyholder feels protected by the early default regulation imposed by the regulator, the contract is more likely closed by the regulator.
instead of the policyholder himself. The second interesting observation is that as volatility $\sigma$ increases the region $\bar{\rho} = 0.3$ expands when there is no early default regulation, while it shrinks when regulator specifies an early termination barrier during the life time of the contract. It indicates that the policyholder becomes more sensitive and exercises the surrender option more likely due to endogenous reasons as the financial world becomes more volatile when no early termination mechanism is established by the regulator. However, when the early termination mechanism is introduced, due to limited rationality, the policyholder is more likely relying on the protection from the regulator as the financial world becomes more volatile so that the region $\bar{\rho} = 0.3$ shrinks as $\sigma$ increases, and at the same time, the early default region expands.

Figure 7: Separating boundaries of the policyholder’s surrender behavior for different investment strategies represented by $\sigma$ when there is the early default intervention, $\theta = 0.9$

5 Conclusion

In this paper we study the impacts of early default risk and surrender risk on the price of participating life insurance policies. The early default of an insurance company is triggered once the company’s asset value touches a predetermined liquidation threshold. The surrender
risk is represented by the surrender intensity which is bounded from below and from above, accounting for the bounded rationality of a representative policyholder in making his surrender decision. The lower bound refers to the policyholder’s surrender intensity for exogenous reasons while the upper bound is achieved if the surrender value is higher than the contract value. Since the early default intervention by the regulator reforms the contract’s payment structure, it influences the policyholder’s surrender behavior, which consequently affects the contract value. We endogenize the policyholder’s surrender in this paper and incorporate the influence of the regulator’s early default intervention on the policyholder’s surrender decision making into the contract valuation. We derive the pricing partial differential equation for characterizing the contract value and solve it numerically with the finite difference method. First of all, given the insurance company’s investment strategy is known, we analyze the influences of the early default risk on the contract’s fair value and consequently on the policyholder’s surrender decision making. Then we discuss the insurance company’s reaction to the regulator’s intervention in terms of its investment strategy and analyze the impacts of its investment strategy on the contract valuation as well as on the policyholder’s surrender behavior.

The analysis for a representative policyholder can be transferred to a large pool of policyholders. Many implications can be drawn from our analysis. First, if policyholders are able to surrender their participating policies optimally, it is not necessary for the regulator to set a regulatory rule to monitor the insurance company. In this case, the insurance company is actually monitored by the policyholders themselves. However, since policyholders are mostly not fully rational to make financially optimal decisions, an early default regulatory rule protects these policyholders as long as the regulation is not too strict. Harsh regulation is disadvantageous to most policyholders. Second, enhancing the regulation exerts a negative effect on the rationality premium for those policyholders who would have more freedom to construct their own optimal stopping strategies if there were fewer regulatory constraints. Since the effects of the early default regulation are different for the policyholders with different surrender reasons and the contract value that we are looking for is the average contract value for all the policyholders, which is denoted as the contract value for a representative policyholder in our paper, the overall effect of enhancing the regulation on the contract value may be positive. Third, in the absence of the early default intervention, we are not clear about the insurance company’s risk preference towards its investment strategy. We find that the equity holder prefers to adopt a less risky investment strategy if the policyholders are able to surrender the contracts optimally. Since the equity holder knows that policyholders are most of the time not financially rational enough and there are always exogenous reasons for them to surrender the contracts prematurely, the equity holder actually tends to invest more riskily. However, when the early default barrier is set, an increase in the riskiness of the investment strategy will generally have a positive effect on the
contract value. The equity holder will then have the incentive to reduce the riskiness of their investment, which is independent of the rationality level of the policyholders. This result is consistent with the goal of the regulator.

Appendix A: Proof of Corollary 1

We follow the proof of Theorem 2.1 in Freidlin (1985) for a similar Dirichlet problem. Define

\[
\begin{align*}
  g(t, A_t) &:= \mu(t)\Psi(t, A_t) + \gamma(t, A_t)S(t, A_t), \\
  c(t, A_t) &:= r(t) + \mu(t) + \gamma(t, A_t), \\
  Y^A_t &:= -\int_0^t c(z, A_z)dz, \\
  U^A_t &:= v(t, A_t)e^{Y^A_t}.
\end{align*}
\]

According to the Ito’s Lemma, we obtain, for all \( t < m < \tau_b \land T \) where \( t < \tau_d \land \tau_s \land \tau_b \land T \), the stochastic differential equation of \( U^A_m \) as

\[
\begin{align*}
  dU^A_m &= \frac{\partial U^A_m}{\partial m}dm + \frac{\partial U^A_m}{\partial A_m}dA_m + \frac{1}{2}\frac{\partial^2 U^A_m}{\partial A_m^2}dA_m^2 \\
  &= \left[ \frac{\partial v}{\partial m}(m, A_m)e^{Y^A_m} - v(m, A_m)e^{Y^A_m}c(m, A_m) \right] dm \\
  &\quad + \frac{\partial v}{\partial A_m}(m, A_m)e^{Y^A_m}(r(m)A_m dm + \sigma(m, A_m)A_m dW_m) \\
  &\quad + \frac{1}{2}\frac{\partial^2 v}{\partial A_m^2}(m, A_m)e^{Y^A_m}\sigma^2(m, A_m)A_m^2dm \\
  &= e^{Y^A_m}(L_v(m, A_m) - c(m, A_m)v(m, A_m))dm + e^{Y^A_m}\frac{\partial v}{\partial A_m}(m, A_m)\sigma(m, A_m)A_m dW_m \\
  &= e^{Y^A_m}(-g(m, A_m))dm + e^{Y^A_m}\frac{\partial v}{\partial A_m}(m, A_m)\sigma(m, A_m)A_m dW_m.
\end{align*}
\]

The last equation follows from equation (15) in Proposition 1.

Integrate both sides of the above equation from \( t \) to \( \tau_b \land T \) and take the expectation on both sides. Under the assumption that

\[
\mathbb{E}_Q \left[ \int_0^{\tau_b \land T} ||e^{Y^A_m}\frac{\partial v}{\partial A_m}\sigma(m, A_m)A_m||^2 dm \right] < \infty
\]
which ensures
\[
\mathbb{E}_Q \left[ \int_t^{\tau_{b,T}} e^{Y_{m}^A} \frac{\partial v}{\partial A_m}(m, A_m) \sigma(m, A_m) A_m dW_m \bigg| G_t \right] = 0,
\]
we obtain
\[
\mathbb{E}_Q[U^A_{\tau_{b,T}} | G_t] = U^A_t + \mathbb{E}_Q \left[ \int_t^{\tau_{b,T}} e^{Y_{m}^A} g(m, A_m) dm \bigg| G_t \right],
\]
and thus
\[
v(t, A_t) = \mathbb{E}_Q \left[ \int_t^{\tau_{b,T}} e^{Y_{m}^A - Y_{t}^A} g(m, A_m) dm + e^{Y_{m}^A - Y_{t}^A} v(\tau_b \wedge T, A_{\tau_{b,T}}) \bigg| G_t \right].
\]
Since
\[
e^{Y_{m}^A - Y_{t}^A} v(\tau_b \wedge T, A_{\tau_{b,T}}) = 1_{\{ \tau_b < T \}} e^{Y_{m}^A} v(\tau_b, A_{\tau_b}) + 1_{\{ \tau_b \geq T \}} e^{Y_{t}^A} v(T, A_T)
\]
\[
= 1_{\{ \tau_b < T \}} e^{Y_{\rho}^A} \Upsilon(\tau_b, A_{\tau_b}) + 1_{\{ \tau_b \geq T \}} e^{Y_{\beta}^A} \Phi(A_T),
\]
where the last equation results from the boundary conditions in Proposition 1, we obtain, by substituting \(g(\cdot, \cdot), c(\cdot, \cdot),\) and \(Y\) with their original forms,
\[
V_t = \mathbb{E}_Q \int_t^{\tau_{b,T}} e^{-\int^t_s (r(u) + \mu(u) + \gamma(u, A_u)) du} (\mu(m) \Psi(m, A_m) + \gamma(m, A_m) S(m, A_m)) dm
\]
\[
+ 1_{\{ \tau_b \geq T \}} \Phi(A_T) e^{-\int^T_t (r(u) + \mu(u) + \gamma(u, A_u)) du} + 1_{\{ \tau_b < T \}} \Upsilon(\tau_b, A_{\tau_b}) e^{-\int^{\tau_b}_t (r(u) + \mu(u) + \gamma(u, A_u)) du} \bigg| G_t
\]
Corollary 1 is therefore proved.

**Appendix B: Proof of Proposition 2**

The pre-death/surrender value function \(v\) is the solution of the PDE (15) with terminal condition \(v(T, A_T) = \Phi(A_T)\) and boundary condition \(v(t, A_t) = \Upsilon(t, A_t)\) for \(A_t \leq B_t\), and bounds \(\rho\) and \(\bar{\rho}\). The pre-death/surrender value function \(w\) is the solution of the same PDE (15) with identical terminal condition \(w(T, A_T) = \Phi(A_T)\) and boundary condition \(w(t, A_t) = \Upsilon(t, A_t)\) for \(A_t \leq B_t\) but different bounds \(\zeta\) and \(\bar{\zeta}\). Assume that \(\zeta \leq \rho\) and \(\bar{\rho} \leq \bar{\zeta}\). Now define \(z = w - v\). It follows directly that \(z(T, A_T) = w(T, A_T) - v(T, A_T) = \Phi(A_T) - \Phi(A_T) = 0\) and \(Z(t, A_t) = \Upsilon(t, A_t) - \Upsilon(t, A_t) = 0\) for \(A_t \leq B_t\). To obtain the dynamics of \(z\) take the
difference of the PDEs describing $w$ and $v$, i.e.:

$$0 = Lw(t, A_t) + \mu(t)\Psi(t, A_t) + \gamma^w(t, A_t)S(t, A_t) - (r(t) + \mu(t) + \gamma^w(t, A_t)) w(t, A_t)$$

$$- (Lv(t, A_t) + \mu(t)\Psi(t, A_t) + \gamma^v(t, A_t)S(t, A_t) - (r(t) + \mu(t) + \gamma^v(t, A_t)) v(t, A_t))$$

$$= Lz(t, A_t) + (\gamma^w(t, A_t) - \gamma^v(t, A_t)) (S(t, A_t) - w(t, A_t)) - (r(t) + \mu(t) + \gamma^v(t, A_t)) z(t, A_t),$$

where $\gamma^v$ and $\gamma^w$, respectively, are given by (7) using the appropriate bounds. Similar to the proof of Corollary 1, we obtain the stochastic representation of $z$ as follows:

$$z(t, A) = \mathbb{E}_Q^{t, A} \left[ \int_t^{\tau_b \wedge T} e^{-\int_t^u (r(u) + \mu(u) + \gamma^v(u, A_u)) du} (\gamma^w(m, A_m) - \gamma^v(m, A_m)) (S(m, A_m) - w(m, A_m)) dm \bigg| \mathcal{G}_t \right],$$

where $\mathbb{E}_Q^{t, A}$ denotes the expectation conditioned on $A_t = A$. From the definition of $\gamma^w$ in (7) and the assumption $\bar{\zeta} \geq \bar{\rho}$ we see that if $(S - w) \geq 0$ we have $\gamma^w = \bar{\zeta} \geq \bar{\rho} \geq \gamma^v$ and thus $(\gamma^w - \gamma^v) \geq 0$. On the other hand, if $(S - w) < 0$ then $\gamma^w = \zeta$. By assumption we have $\zeta \leq \rho$ and thus $\gamma^w \leq \rho \leq \gamma^v$, or, $(\gamma^w - \gamma^v) \leq 0$. Thus, we see that the integrand in the above equation is nonnegative and therefore $z \geq 0$. Since $z = w - v$ we obtain $w \geq v$.

References


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