Contests with Endogenous and Stochastic Entry*

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(Job Market Paper)
Oct, 2011

Abstract

This paper studies imperfectly discriminatory contests which involve costly and endogenous entries. A fixed pool of potential bidders can enter a contest to compete for an indivisible prize. Each bidder incurs an irreversible fixed cost if he decides to enter the contest. After entering, the bidders then bid for the prize. This setting leads to a two-dimensional discontinuous game (Dasgupta and Maskin, 1986). We establish that a symmetric equilibrium exists in the entry-bidding game, where all potential bidders enter with a probability. We further identify the conditions for the existence (non-existence) of a symmetric equilibrium with pure-strategy bidding after entry. Based on the equilibrium result, we explore three main issues about optimal contest design. First, we investigate how the level of accuracy in the winner selection mechanism (i.e. the level of discriminatory power in Tullock rent-seeking contests) affects the expected overall bid. We find the relationship is non-monotonic. A contest designer may benefit from noisier contests, which elicit the optimal overall bid. Second, we study whether a contest designer should exclude potential bidders. Our analysis reveals that she prefers to limit the number of potential bidders by inviting only a subset of them for participation. Finally, we establish that there is no loss of generality to consider contests with nondisclosure of number of actual contestants for optimal design.

JEL Nos: C7, D8.

Keywords: Contest Design; Endogenous Entry; Entry Cost; Stochastic Entry

*An earlier version of this paper was presented at 2010 SAET Meetings under the title of “The Optimal ‘Opaque’ Contests”. We are grateful to Atsu Amegashie, Masaki Aoyagi, Helmut Bester, Oliver Gürtler, Todd Kaplan, Dan Kovenock, John Morgan, Johannes Münster, Aner Sela, Roman Sheremeta, Randy Silvers, Ching-jen Sun, Samarth Vaidya, Cédric Wasser, Elmar Wolfstetter, participants of 2010 SAET Meetings and 2011 International Conference on Contests, Tournaments and Relative Performance Evaluation, and seminar participants at Deakin University and Free University of Berlin for helpful comments and suggestions. All errors remain ours.

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1 Introduction

Economic agents are often involved in contests. They expend costly effort to compete for a limited number of prizes, while their investments are usually non-refundable whether they win or lose. A wide variety of economic activities exemplify such competitions. They include rent-seeking, lobbying, political campaigns, R&D races, competitive procurement, college admissions, ascents of organizational hierarchies, and movement in internal labor markets. The vast wealth of literature on contests has delineated economic agents’ strategic behaviors in contests from diverse perspectives, and has identified the various institutional elements in contest design that affect bidding incentives.

Most existing studies focus on a setting where a fixed number (\(n\)) of bidders participate. These studies, under the “fixed-\(n\) paradigm”, typically abstract away from the ex ante contest participation decisions of bidders and focus on their post-entry activities, assuming that the actual number of active participants is commonly known. In this paper, we complement these studies by explicitly examining a setting where bidders have to make a strategic decision about participating in a contest. They enter contests randomly, so the actual number of participants in a particular contest is uncertain. Participants take into account this uncertainty when placing their bids.\(^1\)

As noted by Konrad (2009), a bidder often bears a nontrivial (fixed) entry cost, which can be explicitly sunk resources or foregone outside opportunities. Incurring the costs allows a bidder to merely participate and is unrelated to their chances of winning.\(^2\) In our setting, a fixed pool of potential bidders decide whether to participate and then sink their bids after entering the contest. Each bidder weighs his expected payoff in future competitions against the entry cost, and participates if and only if the former (at least) offsets the latter. With nontrivial entry costs, we show that a symmetric mixed-strategy equilibrium emerges: each potential bidder enters with the same probability, and adopts the same (possibly mixed) bidding strategy upon entry.

This entry-bidding game complements and enriches the existing literature in several aspects. We elaborate upon its distinct flavors as follows.

First, the strategy of each potential bidder involves two elements in a contest with endogenous entry: (1) whether to enter; and (2) how to bid after entering. This entry-bidding game exemplifies a discontinuous game with two-dimensional actions (Dasgupta and Maskin,

\(^1\)We show later that the optimal contest in general entails nondisclosure of the actual number of contestants.

\(^2\)An analogy is that while an air ticket enables the American tennis player Venus Williams to arrive at the Australian Open, it does not help her win the championship. Similarly, to participate in a R&D tournament, a research company may need to acquire some necessary laboratory equipment to gather project-specific information, or to turn down other profitable tasks, while its chances of winning depend on its subsequent creative input.
The game distinguishes itself from standard contests that are typically identified as uni-dimensional discontinuous games (Baye, Kovenock and de Vries 1994 and Alcalde and Dahm, 2010), where a player’s strategy involves only his bidding action. Due to stochastic entry, the conventional approach to establish equilibrium existence in contests (Baye, Kovenock and de Vries 1994 and Alcalde and Dahm, 2010) does not encompass our settings where the number of active players is uncertain. This novel setting entails the application of Dasgupta and Maskin’s (1986) general theorem on multi-dimensional discontinuous games, which allows us to establish the existence of a symmetric mixed-strategy equilibrium in the entry-bidding game. To our knowledge, our analysis provides the first application of the existence theorem for multi-dimensional settings in the contest literature.

Second, the bidding behavior in contests with stochastic participation has yet to be explored thoroughly. It is well-known in the literature that a bidder’s payoff maximization problem becomes irregular when the contest success function is excessively elastic to effort, e.g. when the discriminatory parameter \( r \) in a Tullock contest exceeds certain boundaries. Stochastic entries further complicate the analysis. By taking into account the uncertainty caused by the stochastic entries, a participant chooses his bid to maximize his expected payoff, which amounts to a weighted sum of a series of irregular functions of his bid. Furthermore, the weights of the summation are determined by the endogenously formed entry probabilities. The general property of a bidder’s overall expected payoff function cannot be readily discerned. We establish sufficient conditions under which participating bidders do (or do not) randomize their bids upon entry. This result allows us to derive an equilibrium bidding strategy in this game and further analyse the design of contests.

Third, endogenous entry yields rich implications for contest design. We follow the mainstream literature by searching for mechanisms that maximize the expected overall bid in a

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3The literature on contests recognizes that (Baye, Kovenock and de Vries 1994, and Alcalde and Dahm, 2010), a well-defined contest success function (e.g., Tullock contest) can be discontinuous at its origin, i.e., when all bidders bid zero.

4To solve for the entry-bidding equilibrium, the traditional approach in the auction literature proceeds in two steps. First, the existence of symmetric bidding equilibrium is shown for each given (symmetric) entry probability and solved for the bidders’ equilibrium payoffs. Second, a break-even condition characterizes the equilibrium entry. This approach is inappropriate in our setting. The first step (finding the bidding equilibrium when potential bidders enter with fixed probabilities) is solvable in an auction setting, but not when the parameter \( r \) for the Tullock contest is big, as in our case. As such, existing results on the existence of equilibria in contests does not apply to contests with random entry and an uncertain number of active players. More detail is provided in Section 3.

5One should note that our two-dimensional strategy space of (entry, effort) cannot be reduced to a setting with single dimensional strategy of effort with a positive fixed cost. In our two dimensional setting, if no one enters the contest, no one wins. If everyone enters but exerts zero effort, every one incurs an entry cost and has an equal chance at winning. In the single dimensional setting, if everyone exerts zero effort, no one incurs any costs but has an equal chance at winning.
contest, and examine three issues: (1) whether the contest designer prefers a more precise winner selection mechanism; (2) whether the contest designer should exclude potential bidders, and invite only a subset of them to participate in the competition; and (3) whether the contest designer could improve the contest’s design by disclosing the actual number of participating bidders when she can observe it.

- **Precision Could Hurt:** We focus on Tullock contests and regard the discriminatory parameter $r$ as a measure of the level of noise in the winner selection mechanism. A greater $r$ implies that a higher bid can be more effectively translated into a higher likelihood of winning, thereby increasing the marginal return to the bid. Conventional wisdom informs us that a greater $r$ provides higher-powered incentives and intensifies competition. We demonstrate, nevertheless, that in our setting the expected overall bid does not vary monotonically with the size of $r$. A contest with a smaller $r$ can paradoxically elicit more effort. An immediate trade-off is triggered when $r$ is raised. A more precise contest incentivizes each participant to bid more, while an overheated competition leaves lesser rent for participants, thereby discouraging entries. Moreover, contestants’ entry probabilities affect the expected overall bids ambiguously. More active entry expands the contest and tends to amplify the overall supply of bids; while it also leads individual participants to bid more prudently, as they anticipate more potential competitors and a smaller chance of winning. The optimum has to balance out these diverse and possibly conflicting forces.

- **Exclusion Helps:** Based on results on optimal precision, we investigate whether the contest designer is better off when there is a larger pool of potential bidders. Without endogenous entry, the contest literature states that the overall bid always increases with the number of bidders. However, our analysis reveals the opposite: contests elicit lesser effort, when a larger pool of potential bidders may enter. Contest designers prefer to limit competition by inviting only a subset of them for participation. The existing studies on shortlisting and exclusion, e.g. those of Baye, Kovenock and de Vries (1993), Taylor (1995), Fullerton and McAfee (1999), and Che and Gale (2003), usually focus on heterogeneous contestants, and concern themselves with selecting (usually two) players of the “right types.” Our result, however, espouses the merit of exclusion in a setting of homogenous players and concerns itself with creating a contest of the “right size.”

- **Opaqueness May Pay Off:** We establish that there is no loss of generality when considering the optimal design of contests that do not disclose the actual number of participants. It is in general suboptimal for the contest designer to announce the actual number of participants when she can observe it.

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6Optimal $r$ is contingent on number of potential contestants.
The rest of the paper proceeds as follows. In Section 2, we discuss the relation of our paper to the relevant literature in the rest of this section. In section 3, we set up the model, and establish our main results on equilibrium existence. Optimal contest design is explored in Section 4, and Section 5 concludes the paper.

2 Relation to Literature

Our paper complements the literature on contests and auctions in various aspects. We next discuss the links to these two strands of literature respectively.

2.1 Contests

Our paper provides a comprehensive and formal account of equilibrium existence in the entry-bidding game. Our paper primarily belongs to the literature on equilibrium existence in contests. Szidarovszky and Okuguchi (1997) establish the existence of pure-strategy equilibria when contestants have concave production functions. The existence and properties of the equilibria remain a nagging problem for contests with less well-behaved technologies. Baye, Kovenock and de Vries (1994) establish the existence of mixed-strategy equilibria in two-player Tullock contests with $r \geq 2$. Alcalde and Dahm (2010) further the literature by showing that all-pay auction equilibria exist under a wide class of contest success functions. Both studies apply the results of Dasgupta and Maskin (1986) on uni-dimensional discontinuous games. Our paper contributes to this literature by introducing bidders’ entry decisions while allowing the number of active bidders to be stochastic. These new flavours enrich our analysis by forming a two-dimensional discontinuous game, and provide a novel application of the general result of Dasgupta and Maskin (1986) on multi-dimensional discontinuous games in the contest literature.

The literature on contests with endogenous entry remains scarce. Higgins, Shughart, and Tollison (1985) in their pioneering work study a tournament model in which each rent seeker bears a fixed entry cost, and randomly participates in equilibrium. In an all-pay auction model, Kaplan and Sela (2010) provide a rationale for entry fees in contests. Besides the differing modeling choice and the diverging focus, Kaplan and Sela (2010) differ from the literature on standard oligopolistic competition. Our paper echoes the argument of Dixit and Shapiro (1986) and Shapiro (1989) on firms’ behavior in oligopolistic markets. He shows that Bertrand competition, which is fiercer, can be more anti-competitive than Cournot competition, which is more subdued, as the latter limits the contestability of the market and discourages entries. We focus on the issue of mechanism design in our particular context. In addition, the level of post-entry competition is a continuous variable and is considered as a strategic choice of the contest designer.

7Our paper can also be related to the literature on standard oligopolistic competition. Our paper echoes the argument of Dixit and Shapiro (1986) and Shapiro (1989) on firms’ behavior in oligopolistic markets. He shows that Bertrand competition, which is fiercer, can be more anti-competitive than Cournot competition, which is more subdued, as the latter limits the contestability of the market and discourages entries. We focus on the issue of mechanism design in our particular context. In addition, the level of post-entry competition is a continuous variable and is considered as a strategic choice of the contest designer.

8Wang (2010) also characterizes the equilibria in two-player asymmetric Tullock contests when $r$ is large.
current paper in a few other aspects. First, they allow players to bear privately-known entry costs, while we assume that entry cost is uniform and commonly known. Second, they let participants know who else has entered, while we focus mainly on uninformed participants. However, we also study the ramifications of disclosure policy as an institutional element of contests.

Two recent experimental studies, Cason, Masters and Sheremeta (2010) and Morgan, Orzen and Sefton (2010), also contribute to this research agenda by studying bidders’ entries. Similar to Morgan, Orzen and Sefton’s (2010) theoretical model, Fu and Lu (2010) also assume that potential bidders enter sequentially, so neither setting involves stochastic participation.

A handful of papers have examined contests with stochastic participation. The majority of these studies, however, assume exogenous entry patterns. Myerson and Wärneryd (2006) examine a contest with an infinite number of potential entrants, whose entry follows a Poisson process. Münster (2006), Lim and Matros (2009) and Fu, Jiao and Lu (2010) assume a finite pool of potential contestants, with each contestant entering the contest with a fixed and independent probability.

The current study also contributes to the growing literature on contest design by exploring the optimal mechanism in a context with endogenous and stochastic entries.

First, our analysis complements the literature on the proper level of precision in evaluating bidding performance. Conventional wisdom says that a precise contest incentivizes aggressive bidding. A handful of studies, however, espouse low-powered incentives in contests and demonstrate that a less “discriminatory” contest can improve efficiency. One salient example is provided by Lazear (1989), who argues that excessive competition leads to sabotage. A more popular stream in the literature instead stresses the “handicapping” effect of the imprecise performance evaluation mechanism in (two-player) asymmetric contests. When contestants differ in their abilities, a noisier contest balances the playfield. This effect encourages weaker contestants to bid more intensely, and deters the stronger ones from shirking. O’Keefe, Vis- cus and Zeckhauser (1984) are among the first to formalize this logic. This rationale is further elaborated upon by Che and Gale (1997, 2000), Fang (2002), Nti (2004), Amegashie (2009), and Wang (2010). In a recent study, Epstein, Mealem and Nitzan (2011) contend that contest designers still prefer all-pay auctions to Tullock contests if they can strategically discriminate between bidders. In contrast to these studies, our paper adopts a N-player symmetric contest, and stresses the trade-off between ex post bidding incentives and ex ante entry incentives. Our paper is closely related to Cason, Masters and Sheremeta’s (2010) experimental study in this aspect, which compares endogenous entries in all-pay auctions and lottery contests.

Our finding on efficient exclusion echoes a handful of pioneering studies by Baye, Kovenock and de Vries (1993), Taylor (1995), Fullerton and McAfee (1999), and Che and Gale (2003). These studies typically involve heterogeneous bidders and identify the subset of bidders with
the most desirable characteristics. Dasgupta (1990) studies a two-stage procurement tournament. Bidders invest in cost reduction in the first stage, and place their bids in the second. Wider competition may diminish bidders’ incentives to engage in R&D. Limiting the number of competing firms may or may not benefit the principal. None of these studies involves entry cost and endogenous entry. In contrast to these studies, an invited (potential) bidder in our setting has to decide whether to enter the subsequent contest, and the entry pattern in the equilibrium remains endogenous and stochastic.

Our study is also related to the literature on efficient disclosure and feedback rules in contests. Lim and Matros (2009) are the first to examine the issue of disclosing the number of contestants, where potential bidders enter with an exogenous probability. They demonstrate the independence of prevailing policy in Tullock contests with $r = 1$ and linear effort costs. Fu, Jiao and Lu (2010) further reveal that the optimal disclosure policy depends on the characteristics of the production functions of contestants. The current paper illustrates the critical role played by the convexity of the bidding cost function and endogeneity of entry.

2.2 Auctions with Stochastic Entry

Our paper is also related to the literature on auctions with endogenous entry. Myerson (1981) shows that a second-price or first-price auction with an optimal reserve price is revenue-maximizing when bidders bear zero entry costs. Samuelson (1985), Menezes and Monteiro (2000) and Lu (2009) require that bidders sink entry costs to participate in auctions. Levin and Smith (1994), Shi (2009), Lu (2010) and Moreno and Wooders (2010) allow bidders to make costly investments to learn their valuations of the object for sale. These studies conclude that revenue maximization requires weaker incentives, i.e. lower reservation prices, than that in Myerson (1981), due to the trade-off between the $ex post$ incentive to bid and the $ex ante$ incentives of entry or information acquisition.

Our study departs subtly from the auction literature in two main aspects. First, the auction design problem addresses an adverse-selection problem: bidders possess private information about their own types and therefore the optimal mechanism screens heterogeneous bidders. Our contest design problem nevertheless concerns itself primarily with a moral hazard problem: the type of player is commonly known, while the optimal mechanism sets out to incentivize effort supply. Second, the auction literature shows that a weaker $ex ante$ incentive, i.e. a reserve price lower than Myerson’s (1981) zero-entry-cost benchmark, is always necessary whenever entry or information acquisition is costly. By way of contrast, the optimum in our setting could involve either a weaker (i.e. a smaller precision $r$) or a stronger (i.e. a bigger

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9Aoyagi (2010), Gershkov and Perry (2009), Ederer (2010), Gürtler and Harbring (2010), and Goltsman and Mukherjee (2011) focus on interim performance feedback in dynamic contests. In contrast, our paper looks at interim feedback on entries.
precision \( r \) \( \text{ex ante} \) incentive than that for the zero-entry-cost benchmark.

Shortlisting and exclusion have long been recognized as an important element in designing auctions with costly entry. Our setting resembles that of Samuelson (1985) and Lu (2009), as both studies assume that bidders bear common entry costs, although the results differ. While Lu (2009) finds that shortlisting is not necessarily optimal, we find that contest designers can always elicit higher overall bids by excluding potential bidders. Levin and Smith (1994) let potential bidders make costly investments to discover their valuations of the object. They establish that the revenue in the optimum decreases with the number of potential bidders to the extent that the information acquisition costs lead to a mixed-strategy entry. Our finding echoes that of Levin and Smith (1994), despite the different settings.

The optimal disclosure policy has also been examined in auctions with a stochastic number of bidders. McAfee and McMillan (1987) and Levin and Ozdenoren (2004) consider exogenous stochastic entry and show that the expected revenue is independent of the disclosure policy when bidders are risk neutral. Our paper allows for endogenous entry and concludes that concealment may elicit strictly higher overall bid under various circumstances.

3 Model and Analysis

In this section, we first set up the model and then conduct the equilibrium analysis.

3.1 Setup

We consider a two-stage game. A fixed pool of \( M(\geq 2) \) identical risk-neutral potential bidders demonstrate interest in a contest with a winner’s purse \( v > 0 \). In the first stage, potential bidders simultaneously decide whether or not to participate. In the second stage, all participants simultaneously submit their bids. A winner is selected and awarded the prize.

3.1.1 Winner Selection Mechanism

Suppose that \( N \geq 2 \) potential bidders enter the contest. They simultaneously submit their bids \( x_i, i = 1, 2, \ldots, N \), to compete for the prize \( v \). The probability of a participating bidder \( i \) winning the prize is given by

\[
p_N(x_i, \mathbf{x}_{-i}) = \frac{x_i^r}{\sum_{j=1}^{N} x_j^r}, \quad \text{if } N \geq 2, \quad \text{and} \quad \sum_{j=1}^{N} x_j^r > 0,
\]

which follows the setup of widely adopted Tullock contest success function. If all participants submit zero bids, the winner is randomly picked from the participants. To the extent that
only one bidder enters, he receives the prize $v$ automatically, regardless of his bid. In the event that nobody enters, the designer keeps the prize.

A bid $x_i$ costs a bidder $c(x_i)$, with $c'(\cdot) > 0$ and $c''(\cdot) \geq 0$. For the sake of tractability, we assume that the bidding cost function takes the form $c(x_i) = x_i^\alpha$, with $\alpha \geq 1$.

It should be noted that our main theorem on equilibrium existence in the entry-bidding game applies to contests with more general success functions and cost functions, which will be discussed in more detail later in the paper.

### 3.1.2 Entry

In the first stage of the game, potential bidders simultaneously decide whether to participate in the contest. Each participant has to sink a fixed cost $\Delta > 0$ if he enters. Entry is irreversible, and the cost $\Delta$ cannot be recovered. We impose the following regularity condition on the model.

**Assumption 1** $\frac{v}{M} < \Delta < v$.

The assumption requires that the entry cost $\Delta$ is nontrivial but not prohibitively high. First, no entry is triggered if it costs more than the winner’s purse. Second, the analysis becomes relatively trivial when entry involves little cost, in which case the institutional elements of the contest do not affect bidders’ entry incentives significantly. Under Assumption 1, no equilibria exist where all potential bidders participate in the contest with certainty.

In our main analysis, we assume that each participating bidder does not know the actual number $N$ of participants. This setting leads to a two-dimensional discontinuous game and demands a more sophisticated analysis. Two remarks are in order. First, entry often involves hidden actions, which cannot be readily observed or verified by other parties. Second, one may view the public observability of $N$ as an institutional element, which is to be chosen strategically by the contest designer. In Section 4.3, we assume that the contest designer is able to observe $N$ and choose the disclosure policy of the contest. We show that a contest would in general elicit lesser bids when $N$ is to be disclosed.

### 3.1.3 Some Preliminaries

Before the formal analysis is carried out, we define two cutoff probabilities, which are used repeatedly throughout the analysis.

**Definition 1** Let $\bar{q} \in (0, 1)$ be the unique solution to $(1 - (1 - q)^M)v - Mq\Delta = 0$, and $q_0 \in (0, 1)$ be the unique solution to $(1 - q)^{M-1}v - \Delta = 0$.

Comparing the two cutoffs leads to the following.
Lemma 1 \( q_0 < \bar{q} \).

Proof. See Appendix.

Let us discuss the implications of the two cutoffs briefly, although their implications unfold as the analysis proceeds. The entry-bidding game cannot trigger an equilibrium, where all potential bidders enter with a probability more than \( \bar{q} \): they would otherwise end up with negative expected payoff in the game. In contrast, the cutoff \( q_0 \) defines a lower bound. If there is an equilibrium where all potential bidders enter with a probability less than \( q_0 \), participating bidders must randomize their bids upon entry.

3.2 Existence of Symmetric Equilibrium

A bidder \( i \)'s behavioral strategy is an ordered pair \( (q_i, \mu_i(x_i)) \), where \( q_i \) is the probability he enters the contest, and \( x_i \) is his bid submitted upon entry. We allow him to randomize on his bids. The probability distribution \( \mu_i(x_i) \) depicts his behavioral bidding strategy conditional on his entry. It reduces to a singleton when the participant does not randomize his bid.

Assumption 1 implies that potential bidders play a mixed-strategy in the entry stage. Each participant is uncertain about the actual level of competition when placing his bid. He bids based on his rational belief about others’ entry patterns. The solution concept of a subgame perfect equilibrium would not apply, because participants possess only imperfect information and no proper subgame exists after the entry stage. We simply use the concept of Nash equilibrium to solve the game. An equilibrium is a strategy combination \( M = (q_i(x_i)) \) of all contestants, which requires that the pair strategy \( (q_i, \mu_i(x_i)) \) of each potential bidder \( i \) maximize his expected payoff based on his rational belief and others’ strategy profile \( (q_j, \mu_j(x_j)) \).

We focus on the symmetric equilibrium of the game where all potential bidders play the same strategy \( (q^*, \mu^*(x)) \). As aforementioned, a potential bidder’s payoff can be discontinuous as the contest success function is discontinuous at origin (see Baye, Kovenock and de Vries, 1994, and Alcalde and Dahm, 2010), i.e. when all participants bid zero. The strategy of each player involves two elements. A conventional approach (in auction literature) to establishing the existence of symmetric equilibria proceeds with two steps, which disentangles the two elements in each player’s strategy and simplifies the analysis. In the first step, for each given (symmetric) entry probability, one shows the existence of symmetric bidding equilibrium and solves for the bidders’ equilibrium payoffs. In the second step, a break-even condition characterizes the equilibrium entry probability. This “disentangling” approach loses its bite in our setting.

First, Dasgupta and Maskin’s (1986) theorem on uni-dimensional games cannot be directly applied to games with an uncertain number of players. The existence of a symmetric bidding
equilibrium under a given entry probability $q$ has yet to be established using alternative approaches. Second, similar to contests with deterministic participation, the bidding game may not be directly solvable when the contest success function is excessively elastic, e.g. when the discriminatory parameter $r$ of a Tullock contest is excessively large. As a result, even if an equilibrium exists, it remains difficult to characterize the properties (e.g. continuity and monotonicity) of bidders’ expected payoffs.

This game, however, can be viewed as a two-dimensional discontinuous game (Dasgupta and Maskin, 1986). We apply the general result of Dasgupta and Maskin (1986) for a multi-dimensional strategy space to establish the existence of symmetric equilibria.

Theorem 1. (a) For any $r > 0$, a symmetric equilibrium $(q^*, \mu^*(x))$ exists. In the equilibrium, each potential bidder enters with a probability $q^* \in (0, \bar{q})$ and his bid follows a probability distribution $\mu^*(x)$. (b) Each potential bidder receives an expected payoff of zero in the entry-bidding equilibrium.

Proof. See Appendix.

To our knowledge, Theorem 1 and its proof provide the first application of Dasgupta and Maskin’s (1986) equilibrium existence result on two-dimensional discontinuous games in the literature on contests. A few remarks are in order. First, the equilibrium existence result applies to broader contexts. We explicitly adopt Tullock technologies to economize on our presentation and facilitate subsequent discussion on contest design. However, the proof of the theorem does not rely on the specific properties of Tullock success functions and the particular form of bidding cost functions. The analysis can be readily adapted to contests with more broadly defined success functions, such as those in Alcalde and Dahm (2010), by redefining the discontinuity set slightly. Second, our analysis has yet to provide a more comprehensive account of equilibrium bidding behaviors, which remains one of the central concerns in contest literature. In this entry-bidding game, a participating bidder may randomize on his bid $x_i$ in the equilibrium. We establish the relevant conditions for pure or mixed bidding strategies subsequently.

Before we proceed, it should be noted that multiple entry equilibria exist in the game. With nontrivial entry cost ($\Delta > \frac{v}{M}$ by Assumption 1), there always exist asymmetric entry equilibria, where a subset of potential bidders stay inactive regardless, while the others enter either randomly or deterministically. Throughout this paper, we focus on symmetric entry equilibria for two reasons. First, symmetric equilibria can be arguably viewed as a natural focal point. Second, many asymmetric equilibria that involve only a subset of $M'(< M)$ active players essentially can be analyzed through the symmetric equilibria in a smaller entry-bidding game with a total of $M'(< M)$ potential bidders.
3.3 Existence of Equilibrium with Pure-Strategy Bidding

Suppose that a symmetric equilibrium with pure-strategy bidding exists. Consider an arbitrary potential bidder $i$ who has entered the contest. Suppose that all other potential bidders play a strategy $(q, x)$ with $x > 0$. He chooses his bid $x_i$ to maximize his expected payoff

$$
\pi_i(x_i | q, x) = \sum_{N=1}^{M} C_{M-1}^{N-1}(1 - q)^{M-N} \left[ \frac{x_i^r}{x_i^r + (N - 1)x^r} u - x_i^{\alpha} \right].
$$

Evaluating $\pi_i(x_i | q, x)$ with respect to $x_i$ yields

$$
\frac{d\pi_i(x_i | q, x)}{dx_i} = \sum_{N=1}^{M} C_{M-1}^{N-1}(1 - q)^{M-N} \left[ \frac{r x_i^{r-1} x^r v}{x_i^r + (N - 1)x^r} \right] - \alpha x_i^{\alpha - 1}.
$$

The (pure) bidding strategy in such an equilibrium, if it exists, can be solved for by the first order condition $\frac{d\pi_i(x_i | q, x)}{dx_i} |_{x_i=x} = 0$. The following lemma fully characterizes such an equilibrium if it exists.

**Lemma 2** Suppose that a symmetric equilibrium $(q^*, x^*)$ with pure-strategy bidding exists. In such an equilibrium, entry probability $q^*$ must satisfy

$$
\sum_{N=1}^{M} C_{M-1}^{N-1}q^{*N-1}(1 - q^*)^{M-N} \frac{v}{N} (1 - \frac{N - 1}{N} \frac{r}{\alpha}) = \Delta.
$$

Each participating bidder places a bid

$$
x^* = \left[ \sum_{N=1}^{M} C_{M-1}^{N-1}q^{*N-1}(1 - q^*)^{M-N} \frac{N - 1}{N^2} \frac{r v}{\alpha} \right]^{\frac{1}{2}}.
$$

The expected overall bid of the contest obtains as

$$
x_T^* = M q^* x^* = M q^* \left[ \sum_{N=1}^{M} C_{M-1}^{N-1}q^{*N-1}(1 - q^*)^{M-N} \frac{N - 1}{N^2} \frac{r v}{\alpha} \right]^{\frac{1}{2}}.
$$

**Proof.** See Appendix. ■

Lemma 2 depicts the main properties of a symmetric equilibrium with pure-strategy bidding, if it exists. We call equation (4) the *break-even condition* of the entry-bidding game with pure-strategy bidding. It determines the entry probability $q^*$ in such an equilibrium. The break-even condition leads to the following.

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10It is impossible to have all participating bidders bid zero deterministically in an equilibrium. When all others bid zero, a participating bidder would prefer to place an infinitely small positive bid, which allows him to win the prize with probability one.
Lemma 3 (a) For any \( r > 0 \), there exists a unique \( q^* \in (0, q) \) that satisfies the break-even condition (4). Hence, \( x^* \) is also uniquely determined for the given \( r \) by the break-even condition (4).

(b) The probability \( q^* \) strictly decreases with \( r \).

Proof. See Appendix. ■

Lemma 3 establishes a unique correspondence between \( r \) and \((q^*, x^*)\). The symmetric equilibrium with pure bidding strategy must be unique for each given \( r \), whenever it exists. However, the strategy profile \((q^*, x^*)\) of Lemma 2 may not constitute an equilibrium.

Consider an arbitrary participating bidder’s payoff maximization problem. Suppose that all other bidders enter the contest with a probability \( q \) and bid \( x \) upon entry. The participating bidder \( i \) chooses his bid \( x_i \) to maximize his expected payoff \( \pi_i(x_i|q, x) \) in the contest, which is the weighted sum of \( \pi_i^N(x_i|q, x) = \frac{x_i^r}{x_i^r + (N-1)x} v - x_i^a \) over all possible \( N \), i.e. \( \pi_i(x_i|q, x) = \sum_{N=1}^{M} C_{M-1}^{N-1} (1 - q^*)^{M-N} \pi_i^N(x_i|q, x) \). Note that each individual component \( \pi_i^N(x_i|q, x) = \frac{x_i^r}{x_i^r + (N-1)x} v - x_i^a \) is simply his expected payoff when he enters a contest in which he competes against \( N-1 \) other bidders deterministically and each of them bids \( x \). We graphically illustrate the relation between \( \pi_i^N(x_i|q, x) \) and \( \pi_i(x_i|q, x) \) in Figure 1.

The equilibrium analysis is trivial when \( r \leq 1 \). In that case, each component \( \pi_i^N(x_i|q, x) \) is concave. Maximizing \( \pi_i(x_i|q, x) \) is thus a well-behaved concave program. In this case, the hypothetical equilibrium bid \( x^* \), which is determined by the first order condition and the symmetry condition, must maximize \( \pi_i(x_i|q^*, x^*) \). A strategy profile with all playing \((q^*, x^*)\) must constitute the unique symmetric equilibrium.
When $r$ is large, however, the function $\pi_N^i(x_i\mid q, x)$ is no longer globally concave. It is well-known in the contest literature that maximizing $\pi_i^N(x_i\mid q, x)$ is not a regular program. The irregularity is exacerbated tremendously in our context. First, it is difficult to draw a general conclusion on the properties of the payoff function $\pi_i(x_i\mid q, x)$, which is the weighted sum of a series of not necessarily concave functions. Second, the weights $\sum_{N=1}^M C_{M-1}^{N-1} q^{N-1}(1 - q)^{M-N}$ literally depend on the entry probability $q$, which, however, is determined endogenously in the equilibrium. The existing results on equilibrium existence obtained from contests with deterministic participation cannot be carried over.

In subsequent analysis, we derive the upper (lower) bound of $r$ which guarantees the existence (non-existence) of a symmetric equilibrium with pure-strategy bidding. Recall the unique correspondence between $r$ and $(q^*, x^*)$ (Lemma 3). Consider a contest with a given $r$. Define

$$\tilde{\pi}_i(x_i) = \pi_i(x_i\mid q^*, x^*) = \sum_{N=1}^M C_{M-1}^{N-1} q^{N-1}(1 - q^*)^{M-N} \left( \frac{x_i^r}{x_i^r + (N-1)x^r} v - x_i^a \right), \quad (7)$$

which is a participating bidder $i$’s expected payoff in the contest when all other bidders play the strategy $(q^*, x^*)$, as given by Lemma 2. Clearly, $\tilde{\pi}_i(x_i)$ is continuous on $[0, \infty)$. The strategy profile $(q^*, x^*)$ constitute an equilibrium if and only if $x^*$ is a global maximizer of $\tilde{\pi}_i(x_i)$.

The next result depicts an important property of the payoff function.

**Lemma 4** When $r \in (1, \alpha(1 + \frac{1}{M-2})]$, $x^*$ is the unique inner local maximizer of $\tilde{\pi}_i(x_i)$ over $(0, \infty)$, i.e. $\tilde{\pi}_i(x^*) > \tilde{\pi}_i(x), \forall x \in (0, \infty)$ and $x \neq x^*$. There exists a unique $x_m \in (0, x^*)$ such that $\tilde{\pi}_i(x_i)$ decreases on $[0, x_m]$, increases on $[x_m, x^*]$, and then decreases on $[x^*, \infty)$.

**Proof.** See Appendix. $\blacksquare$

Lemma 4 depicts the property of $\tilde{\pi}_i(x_i)$ when $r$ remains in the range $(1, \alpha(1 + \frac{1}{M-2})]$. We define $\alpha(1 + \frac{1}{M-2})$ as $+\infty$ when $M = 2$. Although it is no longer globally concave, the function still demonstrates the regularity as depicted by Lemma 4. For $x \in (0, \infty)$, the function is locally minimized at $x_m$ and then maximized at $x^* \in (x_m, \infty)$. Hence, $x^*$ is its unique maximizer for $x \in (0, \infty)$.

However, $x^*$ has yet to be established as the global maximizer: the equilibrium requires that the boundary condition $\tilde{\pi}_i(x^*) \geq \tilde{\pi}_i(0)$ hold. Recall that $x^*$ is uniquely determined by (5) for each given $r$. A participating bidder’s expected payoff in the contest when bidding $x^*$, i.e. $\tilde{\pi}_i(x^*)$, would amount to exactly $\Delta$. However, the bidder automatically receives a reserve payoff $(1 - q^*)^{M-1} v$ from the contest by bidding zero: with a probability $(1 - q^*)^{M-1}$, all other potential bidders stay out of the contest, and a rent of $(1 - q^*)^{M-1} v$ will accrue to him.
automatically. Hence, the bidder has an incentive to bid \( x^* \) only if \((1 - q^*)^{M-1}v \leq \Delta\). The implication of this condition is straightforward: bidding \( x^* (>0) \) is rational only if it generates nonnegative additional return (when all others bid \( x^* \)) in excess of the reservation payoff from bidding zero. The condition essentially requires that \( r \) be bounded from above: the contest must leave sufficient rent to contenders and make sure that each bidder is sufficiently rewarded by bidding \( x^* \).

Recall the cutoff \( q_0 \in (0, \bar{q}) \) depicted by Definition 1, which uniquely satisfies \((1-q_0)^{M-1}v = \Delta\). The unique correspondence between \( r \) and \((q^*, x^*)\), as determined by the break-even condition (4), allows us to obtain the following cutoff of \( r \).

**Definition 2** Define \( r_0 \in (\alpha(1 + \frac{1}{M-1}), 2\alpha] \) to be the unique solution to \( \sum_{N=1}^{M} C^{N-1}_{M-1} q_0^{N-1}(1-q_0)^{M-N} \frac{v}{N!(1 - \frac{N-1}{N} q_0)} = \Delta \).

By Lemma 3(b), \( q^* \) is inversely related to \( r \). The condition \((1-q^*)^{M-1}v \leq \Delta\) requires \( q^* \geq q_0 \), which would hold if and only if \( r \leq r_0 \). We then conclude the following.

**Theorem 2** A symmetric equilibrium with pure-strategy bidding does not exist if \( r > r_0 \).

Similar to contests with deterministic participation, pure-strategy bidding cannot be sustained when \( r \) is excessively large. Theorem 2 provides a sufficient condition under which randomized bidding must occur under endogenous entry. When \( r > r_0 \), the strategy profile \((q^*, x^*)\) would not constitute an equilibrium. In that case, a bidder, when bidding \( x^* \), receives \( \tilde{\pi}_i(x^*) = \Delta \). He would strictly prefer to bid zero, because his expected payoff when bidding zero, \((1-q^*)^{M-1}v\), must be strictly more than \( \Delta \). In other words, \( x^* \) is not a part of the best response of player \( i \) to \((q^*, x^*)\). The symmetric equilibria must involve randomized bidding.

By Lemma 4 and Definition 2, we define the following cutoff of \( r \).

**Definition 3** Define \( \bar{r} \triangleq \min(r_0, \alpha \frac{M-1}{M-2}) \).

The previous analysis leads to the following.

**Theorem 3** For each \( r \in (0, \bar{r}] \), the strategy profile \((q^*, x^*)\), as characterized by Lemma 2, constitutes the unique symmetric equilibrium with pure-strategy bidding during the entry-bidding game.

When \( r \) is bounded from above by both \( r_0 \) and \( \alpha \frac{M-1}{M-2} \), a unique symmetric equilibrium with pure-strategy bidding emerges. The condition \( r \in (0, \bar{r}] \) guarantees: (1) that the payoff function \( \tilde{\pi}_i(x_i) \) is well-behaved, in the sense that its curve reaches a unique peak at \( x^* \) for
and (2) that the boundary condition $\tilde{\pi}_i(x^*) \geq \tilde{\pi}_i(0)$ is met. We then conclude that $x^*$ is the global maximizer when $r$ is subject to both upper bounds. The strategy profile $(q^*, x^*)$ is established as the unique symmetric equilibrium with pure-strategy bidding accordingly.

Theorem 3 imposes a (conservative) upper limit on $r$ for the existence of such an equilibrium. It should be noted that $r \leq \alpha(1 + \frac{1}{M-2})$ is sufficient but not necessary to establish $x^*$ as the local maximizer of $\tilde{\pi}_i(x_i)$ for $x_i > 0$. Analytical difficulty prevents us from fully characterizing the property of $\tilde{\pi}_i(x_i)$ when $r$ exceeds $\alpha(1 + \frac{1}{M-2})$. It remains less than explicit how the equilibrium would behave if $\alpha(1 + \frac{1}{M-2}) < r_0$ and $r \in (\alpha(1 + \frac{1}{M-2}), r_0]$. More definitive conclusions can be drawn in more specific contexts with small numbers of potential bidders.

**Corollary 1** When $M$ is small, i.e. $M = 2, 3$, a symmetric equilibrium with pure-strategy bidding exists if and only if $r \leq r_0$.

In these instances, $(\alpha(1 + \frac{1}{M-2}), r_0]$ is empty, because $\alpha(1 + \frac{1}{M-2}) > r_0$ regardless of $v$ and $\Delta$. Whenever $r$ falls below $r_0$, it automatically satisfies the condition $r \leq \alpha(1 + \frac{1}{M-2})$, which guarantees that $x^*$ maximizes $\tilde{\pi}_i(x_i)$.

However, technical difficulty prevents us from drawing more general conclusions analytically when $M$ is larger, which may lead $r_0$ to exceed $\alpha(1 + \frac{1}{M-2})$. We resort to a numerical exercise to obtain further insights about the properties of the expected payoff function $\tilde{\pi}_i(x_i)$. For expositional efficiency, we postpone the presentation and discussion of these observations to Section 4.1.2 as they shed light on the optimal contest design problem explored in that section.

## 4 Contest Design

The equilibrium behavior of the bidders may depend critically on the institutional elements of the contest. Central to the contest literature is the question of how the contest rules affect equilibrium bidding. As Gradstein and Konrad (1999) argued, “... contest structures result from the careful consideration of a variety of objectives, one of which is to maximize the effort of contenders.” Based on the equilibrium analysis, we follow this literature to discuss the optimal design of a contest that maximizes the overall bid. Specifically, we consider three main issues: (1) the optimal level of accuracy of the winner selection mechanism (the proper size of $r$ in Tullock contests); (2) the efficiency implications of shortlisting and exclusion; and (3) the optimal disclosure policy.

### 4.1 Optimal Accuracy: Choice of $r$

In a Tullock contest, the parameter $r$ reflects the “discriminatory power” or the level of precision of the winner selection mechanism in the contest. With a higher $r$, a bidder’s
win depends more on the level of his bid, rather than other noisy or random factors. The level of precision in a contest is largely subject to the autonomous choice of the contest designer. For instance, the designer can modify the judging criteria of the contest to suit her strategic goals, e.g. adjusting the weights of subjective component in contenders’ overall ratings. Alternatively, she can vary the composition of judging committees (experts vs. non-experts).

Following the literature (e.g. Nti, 2004), we let \( r \) be chosen strategically by the contest designer. We then consider a three-stage game. The designer chooses \( r \) and publicly announces it in the beginning. Next, the entry-bidding game takes place. In the subsequent analysis, we investigate how the size of \( r \) affects the equilibrium bids.

Before we proceed, we consider the benchmark case of a contest with a fixed number \( M \) of participants. A larger \( r \) increases the marginal return of a bid and further incentivizes bidders. It is well known in the contest literature that both individual bids and overall bids strictly increase with \( r \) whenever a pure-strategy equilibrium exists, i.e. \( r \in [0, \alpha(1 + \frac{1}{M-1})] \). This conventional wisdom, however, loses its bite in our setting with endogenous entry.

### 4.1.1 Optimum

A contest with endogenous and costly entry involves tremendously more extensive strategic trade-offs. On the one hand, a more discriminatory contest compels participants to bid more, while on the other, the increasing dissipation of rent limits entry. As revealed by Lemma 3, \( q^* \) would strictly decrease with \( r \) in the symmetric equilibrium with pure-strategy bidding.

This trade-off, however, does not exhaust the intricacy involved in the determination of optimal \( r \). An additional trade-off is triggered at a differing layer. More extensive participation (i.e. a higher \( q^* \)) does not necessarily improve the supply of bids in the contest. On the one hand, the contest on average engages more bidders, which amplifies the sources of contribution and tends to increase the overall bid. On the other hand, each participant would bid less, as they anticipate more intense competition and therefore expect lesser reward. The overall effect has yet to be explored more formally.

Consider an arbitrary entry-bidding game where potential bidders enter with a probability \( q^* \in (0, 1) \) in a symmetric equilibrium. The prize \( v \) is given away with a probability \( 1 - (1 - q^*)^M \). Hence, bidders win an expected overall rent of \( [1 - (1 - q^*)^M]v \); while they on average incur entry cost \( Mq^*\Delta \). The following fundamental equality must hold in this symmetric equilibrium:

\[
[1 - (1 - q^*)^M]v \equiv Mq^*(\Delta + E(x^\alpha)).
\]  

The fundamental equality allows us to identify the expected overall bidding cost incurred by the bidders in the equilibrium without explicitly solving for it:

\[
Mq^*E(x^\alpha) = [1 - (1 - q^*)^M]v - Mq^*\Delta.
\]
The convexity of cost function \((\alpha \geq 1)\) further implies that the expected overall bid \((Mq^*E(x))\) must be bounded from above:

\[
(Mq^*E(x)) = Mq^*E[(x^\alpha)^{1\over \alpha}] \leq Mq^*[E(x^\alpha)]^{1\over \alpha}.
\] (10)

By the fundamental equality (8) or (9), we further obtain

\[
(Mq^*E(x)) \leq [Mq^*]^{\alpha - 1\over \alpha}[1 - (1 - q^*)^M]v - Mq^*\Delta)^{1\over \alpha}.
\] (11)

Equation (11) yields important implication: Regardless of the equilibrium bidding strategy upon entry, RHS of (11) imposes an upper limit on the expected overall bids that an equilibrium with entry probability \(q^*\) could elicit. The expected overall bid \((Mq^*E(x))\) reaches the upper limit \([Mq^*]^{\alpha - 1\over \alpha}[1 - (1 - q^*)^M]v - Mq^*\Delta)^{1\over \alpha}\) if and only if: (1) bidders play a pure bidding strategy upon entry; or (2) participants randomize their bids but \(\alpha = 1\).

Denote the upper boundary by

\[
\overline{x_T}(q) \triangleq (Mq)^{\alpha - 1\over \alpha}[1 - (1 - q)^M]v - Mq\Delta)^{1\over \alpha}
\] (12)

with \(q \in (0, 1)\). The function \(\overline{x_T}(q)\) exhibits the following important properties.

**Lemma 5** (a) There exists a unique \(\hat{q} \in (q_0, \tilde{q})\), which uniquely maximizes \(\overline{x_T}(q)\);

(b) The function \(\overline{x_T}(q)\) strictly increases with \(q\) when \(q \in (0, \hat{q})\), and strictly decreases when \(q \in (\hat{q}, 1)\).

**Proof.** See Appendix. ■

As stated by Lemma 5, the function \(\overline{x_T}(q)\) varies non-monotonically with \(q\) and is uniquely maximized by \(\hat{q} \in (q_0, \tilde{q})\). This property implies that the overall bid that can be possibly elicited from the contest will never exceed \(\overline{x_T}(\hat{q})\), regardless of the prevailing contest rules.

**Definition 4** Define \(\overline{x_T^*} \equiv \overline{x_T}(\hat{q})\), which indicates the maximum amount of the overall bid a contest can elicit.

The key to the design problem unfolds in Lemma 5: a mechanism must be optimal if it achieves the “first best” \(\overline{x_T^*}\). We subsequently discuss the possibility of eliciting the “first best” through contest design.

By Lemma 5 and (8)-(11), the first best \(\overline{x_T^*}\) can be achieved in a symmetric equilibrium with an entry probability \(\hat{q}\), if there exists a \(\hat{r}\) that induces entry probability \(\hat{q}\) and (1) participants play a pure bidding strategy upon entry in the equilibrium; or (2) participants randomize their bids but \(\alpha = 1\). For any given \(r\), the exact forms of bidding strategies in equilibria with mixed bidding remain unknown. It is difficult to identify the correspondence
between prevailing contest rules and the subsequent equilibrium when it involves randomized bidding. We thus focus on the possibility of inducing the “first best” in equilibria with pure-strategy bidding. Nevertheless, our investigation shows it is rather sufficient to focus on contests that induce pure-strategy bidding.

Recall that Lemma 3 establishes the unique correspondence between \( r \) and \((q^*, x^*)\) if a symmetric equilibrium with pure-strategy bidding exists. The equilibrium with entry probability \( q^* \) is determined by the break-even condition:

\[
v \sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1} (1 - q^*)^{M-N} \left[ \frac{1}{N} - \frac{N-1}{N^2} \frac{r}{\alpha} \right] = \Delta.
\]

We highlight the following cutoff.

**Definition 5** Let \( r(\hat{q}) \) be the unique solution of \( r \) to

\[
v \sum_{N=1}^{M} C_{M-1}^{N-1} \hat{q}^{N-1} (1 - \hat{q})^{M-N} \left[ \frac{1}{N} - \frac{N-1}{N^2} \frac{r}{\alpha} \right] = \Delta. \tag{13}
\]

The following result is formally stated.

**Theorem 4** (a) \( r(\hat{q}) < r_0 \).

(b) Whenever \( r(\hat{q}) \leq \alpha(1 + \frac{1}{M-2}) \), the contest designer can elicit the “first best” \( x_T^* \) by setting \( r = r(\hat{q}) \). It induces a symmetric equilibrium with pure-strategy bidding. Potential bidders enter the contest with a probability \( \hat{q} \) in the symmetric equilibrium.

**Proof.** See Appendix. \( \blacksquare \)

Setting \( r \) to \( r(\hat{q}) \) could allow the contest designer to elicit the “first best” \( x_T^* \equiv \overline{x_T}(\hat{q}) \). Because \( r(\hat{q}) \in (0, r_0) \), whenever \( r(\hat{q}) \) falls below \( \alpha(1 + \frac{1}{M-2}) \), it satisfies the sufficient condition \( r \leq \overline{r} \), thereby inducing a symmetric equilibrium with pure-strategy bidding by Theorem 3. In the equilibrium, potential bidders enter the contest with a probability of exactly \( \hat{q} \). By Lemma 5, the expected overall bid must strictly decrease when \( r \) deviates from \( r(\hat{q}) \).

Additional discussion is provided as follows to complement our analysis.

### 4.1.2 Discussion

Two main issues are discussed. First, we compare our results to benchmark cases. Second, we examine the robustness of our result to what extend the condition \( r(\hat{q}) \) could robustly induce pure-strategy bidding.

**Comparison to Benchmark Cases** Our results run in sharp contrast to the conventional wisdom in contest literature. In a contest with a fixed number \( M \) of participants or free entry, a higher \( r \) provides stronger incentives to bidders, and elicits strictly higher bids whenever a
pure-strategy equilibrium prevails, i.e. when \( r \leq \alpha (1 + \frac{1}{M-1}) \). The size of \( r \) in our setting, however, affects the resultant equilibrium bid non-monotonically.

Despite the various trade-offs between conflicting forces, a softer \textit{ex ante} incentive, i.e. a smaller \( r \), may or may not be optimal. The optimal size of the parameter could either fall below or remain above the benchmark \( \alpha (1 + \frac{1}{M-1}) \). In the left panel of Figure 2, the observations demonstrate the incidence of optimal “soft” incentives, with \( r(\hat{q}) < \alpha (1 + \frac{1}{M-1}) \). In the right panel, the results illustrate the possibility of the opposite, with \( r(\hat{q}) \in (\alpha (1+\frac{1}{M-1}), \alpha (1+\frac{1}{M-2})) \).

![Figure 2](image)

These observations also contrast the results of related studies in auction literature. A number of studies have been devoted to the optimal design of auctions with costly entry, including Menezes and Monteiro (2000) and Lu (2009), Levin and Smith (1994), Shi (2009), Lu (2010) and Moreno and Wooders (2010). They all espouse the optimality of a “softer” incentive: the optimal reserve price is always lower than in the free-entry benchmark. The insight from auction literature does not extend to our setting. The observations in Figure 2 demonstrate that the optimum does not necessarily requires a lower-powered incentive mechanism than the free-entry benchmark level \( \alpha (1 + \frac{1}{M-1}) \).

Robustness and Numerical Exercises  Our analysis has been limited so far. The global optimality of \( r(\hat{q}) \) is conditioned on that it also leads to pure-strategy bidding. It remains to be explored to what extent \( r(\hat{q}) \) could robustly induce pure-strategy bidding.

Because \( r(\hat{q}) < r_0 \), pure-strategy bidding can be induced as long as \( r(\hat{q}) \) falls below \( \alpha (1 + \frac{1}{M-2}) \). A definitive conclusion can be drawn in contests with small pools of potential participants.

\textbf{Corollary 2}  \textit{When the contest is small, i.e.,} \( M = 2, 3 \), \( r(\hat{q}) \leq \alpha (1 + \frac{1}{M-2}) \) \textit{must hold, and} a

\footnote{In our setting, if entry does not involve fixed entry cost, all the \( M \) potential bidders will participate. The conventional wisdom in contest literature would apply, such that \( r = \alpha (1 + \frac{1}{M-1}) \) would emerge as the optimum.}
symmetric equilibrium with pure-strategy bidding can always be induced by setting \( r = r(\hat{q}) \).

In these cases, \( \alpha(1 + \frac{1}{M-2}) > r_0 \), so the condition \( r(\hat{q}) \leq \alpha(1 + \frac{1}{M-2}) \) is satisfied automatically. Nevertheless, it is less certain when \( M \) is large. We further check its robustness through numerical exercises. The condition is found to hold over a large parameter space, and ample incidents can be observed. Figure 2, which is provided above, gives a small sample of these observations.

We observe incidents of \( r(\hat{q}) > \alpha(1 + \frac{1}{M-2}) \) as well. However, recall that \( r \leq \alpha(1 + \frac{1}{M-2}) \) is sufficient but not necessary for pure-strategy bidding. It should be noted that pure-strategy bidding can still be induced by \( r \in (\alpha(1 + \frac{1}{M-2}), r_0] \). As aforementioned, technical difficulty prevents us from drawing definitive conclusion on the property of bidders’ expected payoff function \( \tilde{\pi}_i(x_i) \) when \( r \) exceeds \( \alpha(1 + \frac{1}{M-2}) \). Our numerical exercises, however, yield interesting observations.

We normalize \( v \) to unity. The simulation is run over a large set of the parameters \((\alpha, M)\), which span the entire space of \([1, 2] \times \{4, 5, \ldots, 100\}\). For given \((\alpha, M)\), we let \( r \) vary over the entire range of \((\alpha(1 + \frac{1}{M-2}), r_0]\) if \( \alpha(1 + \frac{1}{M-2}) < r_0 \), and let \( \Delta \) vary over the interval \((\frac{1}{M}, 1)\) as required by Assumption 1. We observe from our simulation results, with no exception, that all \( \tilde{\pi}_i(x_i) \) demonstrates the property depicted by Lemma 4, and is uniquely maximized by \( x^* \), despite that \( r \) exceeds \( \alpha(1 + \frac{1}{M-2}) \). In all resultant figures, the curve is regularly shaped as described by Lemma 4. Figure 3 provides one example of them. The strategy profile \((q^*, x^*)\) constitutes the unique symmetric equilibrium with pure-strategy bidding in all the simulated settings.

Hence, in all the simulated settings, we can elicit the “first best” by setting \( r = r(\hat{q}) \), regardless of the size of \( r(\hat{q}) \). Based on these observations, we propose the following conjectures.
Conjecture 1  (a) A symmetric equilibrium with pure bidding exists if and only if $r \leq r_0$.

(b) The first best overall bid $\bar{x}^*$ can always be induced in a unique symmetric equilibrium with pure-strategy bidding by setting $r$ to $r(\hat{q})$.

We are unable to prove it analytically. However, all of our numerical exercises lend support to the claim. We leave it to future studies due to its technical difficulty.

4.2 Efficient Exclusion

The equilibrium analysis also allows us to investigate another classical question in the literature on contest design. With a fixed number $n$ of bidders, a Tullock contest elicits an overall bid of $ar^{\frac{n^2}{n-1}}$ whenever a pure-strategy equilibrium exists. The number of overall bids strictly increases with the number of bidders $n$. A handful of studies, including Baye, Kovenock and de Vries (1993), Taylor (1995), Fullerton and McAfee (1999), and Che and Gale (2003), demonstrate that a contest designer may benefit from narrowing the slate of potential prize winners and excluding a subset of contestants. This strand of literature conventionally focuses on heterogeneous players and concerns themselves with selecting bidders of proper types. None of these studies involves stochastic and endogenous entry. In what follows, we demonstrate that exclusion can improve the efficiency of the contest in our setting despite the potential bidders being symmetric.

Consider our basic setting where $M$ potential bidders are interested in participating in the competition. We now allow the contest designer to invite only a subset of these bidders for participation. The invited bidders then decide whether to participate in the contest after they observe the rules of the contest, i.e. the size of $r$ set by the contest designer.

Let $M'$ be an arbitrary positive integer. Define $M_0 \triangleq \min(M' \mid \frac{v}{M'} < \Delta)$ and assume $M_0 < M$. Recall that the amount of overall bid in a given contest is bounded from above by the first best $\bar{x}^*_T$ (see Lemma 5 and Definition 4), which can be achieved when $r$ is set to $r = r(\hat{q})$, and $r(\hat{q})$ leads to pure-strategy bidding. It should be noted that the exact amount of $\bar{x}^*_T$ depends on the number of potential bidders who may enter the contest. Let $\bar{x}^*_T(M')$ be the first best bid for a contest with $M'$ potential bidders. The function $\bar{x}^*_T(M')$ exhibits the following property.

**Lemma 6** $\bar{x}^*_T$ strictly decreases with $M'$ for all $M' \geq M_0$.

**Proof.** See Appendix. ■

Lemma 6 shows that the first best $\bar{x}^*_T$ of a contest strictly declines with $M'$. The result yields direct implications for the contest design: a contest may have a weaker potential of eliciting bid if it involves a larger pool of potential bidders. We allow the contest designer to set $r$ strategically. Let $r(\hat{q}(M'))$ be the unique solution to (13) in a contest with $M'$ potential bidders. We conclude the following.
Theorem 5 Whenever \( r(\hat{q}(M_0)) \leq \alpha(1 + \frac{1}{M_0 - 2}) \), the contest designer will not invite more than \( M_0 \) contestants.

Theorem 5 demonstrates that exclusion improves bidding efficiency. Whenever the condition \( r(\hat{q}(M_0)) \leq \alpha(1 + \frac{1}{M_0 - 2}) \) is met, the contest designer will get strictly better off by excluding \( M - M_0 \) potential bidders from the contest. By inviting \( M_0 \) of them, and setting \( r \) to \( r(\hat{q}(M_0)) \), it elicits an overall bid \( \tilde{x}^*(M_0) \), which, by Lemma 6, is unambiguously more than what she can possibly achieve if she engages a greater number of potential bidders. Our result thus provides an alternative rationale for shortlisting and exclusion in a setting with homogeneous bidders but endogenous entry. The logic resembles that on the optimal \( r \). To put it simply, although inviting more bidders may engage more participants to contribute their bid, each of them would enter less often and bid less (if he enters) anticipating a more intense competition and expecting subsequently a smaller share of the rent. Further, more frequent participation may lead to excessive rent dissipation because of the entry costs incurred, which tends to limit bidders’ effort supply.

Theorem 5 shows that the optimal number of invited bidders must not exceed \( M_0 \). It provides only an upper bound for the possible optimum, and does not pin down exactly how many bidders should be invited in the optimum. When the contest designer invites less than \( M_0 \) potential bidders, the overall bid of the contest can elicit would change indefinitely, and the efficiency of the contest may either improve or suffer.\(^{12}\)

The analysis for a contest with less than \( M_0 \) potential bidders is beyond the scope of the current paper, as Assumption 1 no longer holds in that setting. The alternative context in fact renders an even more handy equilibrium analysis. Most of the analysis in the current setting would not lose its bite after slight alteration. However, a general and systematic conclusion on the exact optimum \( M^* \) remains difficult. First, the optimization problem requires comparison across integers. Second, bidders behave qualitatively differently across the two contexts, i.e., when the number of potential bidder is above and below \( M_0 \). The discontinuity makes the comparison depend sensitively on the specific settings of \((v, \Delta)\), which, in general, does not exhibit regularity.

4.3 Disclosure Policy

Our analysis so far assumes that the actual participation level \( N \) is unknown to bidders. Firms’ actual entry often involves unobservable or unverifiable actions. However, we now consider it as an institutional element. We assume that the actual participation rate is observable to

\(^{12}\)Examples in specific settings are available from the authors upon request, which demonstrate that the overall bid may either decrease or increase.
the contest designer, and we explore whether the designer can benefit from disclosing $N$, i.e. eliciting a higher amount of overall bid.

We let the contest designer commit to her disclosure policy prior to the entry-bidding game. Upon learning the disclosure policy, bidders enter and bid. Denote by $d$ the policy that commits to announcing the true realization of $N$ to participating bidders and by $c$ the policy that conceals the actual $N$. Participants learn $N$ before they bid if and only if policy $d$ is chosen. Under the former, the actual number of participants $N$ becomes common knowledge upon its realization. Under the latter, our benchmark setting remains.

The same issue has also been investigated in other studies that involve stochastic participation. The question is raised in the auction literature by McAfee and McMillan (1987) and Levin and Ozdenoren (2004). Lim and Matros (2009) and Fu, Jiao and Lu (2010) explore this issue in auctions and contests with exogenous stochastic entries.

### 4.3.1 Equilibrium When $N$ is Disclosed

Under policy $d$, the analysis on the entry-bidding game is simplified substantially. Each contest after the entry stage is a proper subgame. With $N$ to be known to participating bidders, each subgame of $N$-person contest boils down to a uni-dimensional game. The existence theorem of Dasgupta and Maskin (1986) for uni-dimensional discontinuous games allows us to verify the existence of equilibrium in every possible subgame. We then readily establish the existence of symmetric equilibria in the entry-bidding game.

**Theorem 6** For any given $r > 0$, there exist symmetric subgame perfect equilibria $(q^*_d, \{x^*_N, N = 1, 2, ..., M\})$ in the entry-bidding game. All potential bidders enter with a probability $q^*_d \in (0, 1)$, and play a (pure or mixed) bidding strategy $x^*_N$ in each subgame with $N$ entrants. Each potential bidder receives zero expected payoff in the entry-bidding game.

**Proof.** See Appendix. ■

As well known in the contest literature, in a given subgame of an $N$-person contest, pure-strategy bidding emerges in the equilibrium if and only if $r \leq \alpha(1 + \frac{1}{N-1})$. Denote by $x^*_T(r, t)$ the expected overall bid in a contest with a discriminatory parameter $r$ and under a disclosure policy $t$. We further obtain the following.

**Lemma 7** Suppose that the contest designer chooses policy $d$, and $r \leq \alpha(1 + \frac{1}{M-1})$. There exists a unique symmetric equilibrium in the entry-bidding game. The equilibrium leads to pure-strategy bidding in all subgames. Each potential bidder enters the contest with a probability $q^*_d \in (0, 1)$, which uniquely solves

$$
\sum_{N=1}^{M} C_{N-1}^{M-1} q^*_d^{N-1} (1 - q^*_d)^{M-N} \pi_N = \Delta,
$$

\(13\) Under policy $c$, the theorem for uni-dimensional game does not apply as the bidding game involves an uncertain number of bidders.
where $\pi_N$ is the payoff of an entrant in a subgame with $N$ entrants. In the equilibrium, the contest elicits an expected overall bid

$$x_T^+(r, d) = M q_d^* \sum_{N=1}^{M} C_{M-1}^{N-1} q_d^{N-1} (1 - q_d^*)^{M-N} \left( \frac{N - 1 - r v}{N^2 - \alpha} \right)^{\frac{1}{2}}.$$  \hfill (15)

**Proof.** See Appendix. \hfill \Box

When $r$ exceeds $\alpha(1 + \frac{1}{M-1})$, mixed-strategy bidding arises in subgames of large $N$.

### 4.3.2 Optimal Disclosure Policy under Pure-Strategy Bidding

The aforementioned existing studies in auction and contest literature are typically based in settings where pure-strategy bidding equilibrium would emerge regardless of the prevailing disclosure policy. To facilitate comparison across the two scenarios, we focus on the setting with $r \leq \alpha(1 + \frac{1}{M-1})$. Under this condition, participating bidders would not randomize over their bids regardless of the prevailing disclosure policy.

The expected overall bid under policy $c$ is simply given by (6). We compare (6) with (15), which leads to the following.

**Theorem 7** For given $r \in (0, \alpha(1 + \frac{1}{M-1}))$, we have $x_T^+(r, c) \geq x_T^+(r, d)$ if and only if $\alpha \geq 1$. That is, concealing the actual number of participating bidders allows the contest to elicit a strictly higher amount of expected overall bids if and only if the bidding cost function is strictly convex. In addition, the resultant expected overall bid of the contest is independent of the prevailing disclosure policy if and only if the bidding cost function is linear.

**Proof.** See Appendix. \hfill \Box

A few remarks are in order. First, the same result would continue to hold when the contest designer is allowed to partially disclose the realization of $N$. That is, she is allowed to disclose the range of $N$ but not its exact realization. We omit it for brevity but the detail is available upon request.

Second, our analysis provides new insights on the well known “disclosure-independence principle” in auction literature (e.g. Levin and Ozdenoren, 2004), and contest literature (Lim and Matros, 2009, and Fu, Jiao and Lu, 2010). As shown in Theorem 7, the resultant expected overall bid is independent of the prevailing disclosure policy if and only if bidding cost is linear, while it depends on the disclosure policy when the bidding cost function is nonlinear. Theorem 7 thus provides another incident of “disclosure-dependence.” The logic of this result parallels that of Fu, Jiao and Lu (2010) in explaining why concealment elicits higher overall bid when the characteristics function is strictly concave. Convex bidding cost leads bidding behavior to exhibit “pseudo risk aversion”. When $N$ is to be disclosed, bidders “over-react” to “unfavorable contests” (i.e. those with large $N$) by reducing their bids substantially, but
“under-react” to “favorable contests” (i.e. those with small $N$) by increasing their bids less than proportionally. Concealment alleviates the adverse effect. More detailed discussion on “pseudo risk aversion” can be seen in Fu, Jiao and Lu (2010).

Third, the analysis has focused on the tractable case of $r \leq \alpha(1 + \frac{1}{M-1})$, such that pure-strategy bidding always arises. It remains to be investigated how the prevailing disclosure policy determines the expected overall bid when $r$ exceeds the cutoff and pure-strategy bidding does not necessarily arise in the equilibrium. A direct comparison between the two schemes is limited by existing techniques in delineating symmetric bidding behavior and the resultant rent dissipation when $r$ is large. Baye, Kovenock and de Vries (1994) demonstrate that rent is fully dissipated in two-bidder contests when $r$ exceeds two. Alcalde and Dahm (2010) characterize the asymmetric equilibrium (all-pay auction equilibrium) and resultant equilibrium rent dissipation in $n$-bidder contest. These findings do not directly shed light on our setting. Furthermore, our analysis is complicated when bidding cost is allowed to be convex. However, the following claim can still be made.

**Remark 1** When $r \in (\alpha(1 + \frac{1}{M-1}), \alpha(1 + \frac{1}{M-2}))$, the “disclosure-independence principle” does not hold even if bidding cost is linear.

**Proof.** See Appendix.

The result imposes a further limit on the scope of the “disclosure-independence principle”: $x_T^*(r, c) \neq x_T^*(r, d)$ even if bidding cost is linear when $r \in (\alpha(1 + \frac{1}{M-1}), \alpha(1 + \frac{1}{M-2}))$. The prevailing disclosure policy does play a role in determining the equilibrium overall bid.$^{14}$

However, when $r > \alpha(1 + \frac{1}{M-1})$, mixed-strategy bidding will be definitely involved in the comparison between the two disclosure policies, which makes it technically challenging to determine the optimal disclosure policy. Nevertheless, we next show that from a perspective of contest design, there is no loss of generality to focus on contests with nondisclosure of number of actual contestants for optimal design.

### 4.3.3 A Broader Perspective: Mechanism Design

Despite the analytical difficulty in comparing $x_T^*(r, c)$ with $x_T^*(r, d)$ directly when $r$ is large, the incompleteness of the direct comparison is of lesser concerns when the issue is examined from the perspective of mechanism design, i.e. when $r$ is allowed to be chosen by the designer.

**Theorem 8** Suppose that $r(\hat{q})$ (as identified in Definition 5) can induce a symmetric equilibrium with pure-strategy bidding under policy $c$. A contest $(r(\hat{q}), c)$ dominates any contest $(r, d)$ regardless of $r$, i.e. $x_T^*(r(\hat{q}), c) \geq x_T^*(r, d)$, $\forall r \in (0, \infty)$.

${^{14}}$Remark 1 is likely to hold for any $r > \alpha(1 + \frac{1}{M-1})$. 

**Proof.** See Appendix. ■

Theorem 8 states that policy $d$ (that discloses the number of participating bidders) would not lead to more efficient bidding when $r$ can be set by the contest designer. The logic underlying Theorem 8, to a large extent, reflects a broad argument from the perspective of mechanism design. It should be noted that the amount of overall bid a contest can possibly elicit can never exceed $\bar{r}_T^*$, regardless of the prevailing disclosure policy. Hence, when a contest $(r(\hat{q}), c)$ can successfully achieve the first best, it must (at least weakly) dominate all other possible mechanisms.

5 Concluding Remarks

In this paper, we provide a thorough account of contests with endogenous and stochastic entries. We show the existence of a symmetric mixed-strategy equilibrium in which potential bidders randomly enter. We also provide a sufficient condition under which participants engage in pure bidding actions. Based on these equilibrium results, we identify relevant institutional elements in contest rules, and we demonstrate that analysis in this setting adds substantially to existing knowledge on optimal contest design.

While our study is one of the first to investigate the subtle and rich strategic interaction that occurs in contests with endogenous entries, our analysis reveals the enormous possibilities for future studies. Due to analytical difficulties, the open conjectures in Section 4 pose a challenge for future research on contests. However, the authors will attempt this, despite the technical difficulties.

Further, our setting (characterized by common entry cost and resultant stochastic entry) is only one way for modeling contests that involve endogenous entry. Other examples include the setting of Kaplan and Sela (2010). They consider all-pay auctions with privately-known entry costs. Another possibility is to allow for non-uniform but commonly-known entry cost. The setting has not been widely studied in contest literature. It would lead to a “stratified” entry pattern, under which a portion of bidders with lower costs participate deterministically while the rest remain inactive. In this case, the cutoff type breaks even while other participants end up with positive rents. The optimal contest rules under this setting deserve more serious exploration, which the authors will also pursue in the future.
Appendix

Proof of Lemma 1

Proof. Let $f_1(q) = [1 - (1 - q)^M]v - Mq\Delta$, and $f_2(q) = (1 - q)^{M-1}v - \Delta$. $q (> 0)$ is defined as $f_1(q) = 0$. The first order derivative of $f_1(q)$ is $f'_1(q) = Mf_2(q)$, which is a decreasing function of $q$. $f'_1(q)$ is positive when $q = 0$, and it is negative when $q = 1$.

$q_0$ is defined as $f_2(q_0) = 0$. Therefore, $f_1(q)$ increases on $[0, q_0]$, and decreases from $[q_0, 1)$. $f_1(q)$ thus has two zero points, i.e. $\{0, q_0\}$, and $q_0 < q$. ■

Proof of Theorem 1

Proof. Part (a) Existence of symmetric equilibria: Consider the following extended game. There are $M$ contestants who simultaneously choose their two-dimensional actions, which are denoted by $a_i = (a_{i1}, a_{i2}) = (q_i, x_i) \in A$, $i = 1, 2, ..., M$, where the uniform action space $A = [0, 1] \times [0, v^{1/\alpha}]$ is nonempty, convex and compact.

Let $k = (k_1, k_2, ..., k_i, ..., k_N)$ where $k_i$ is either 0 or 1. Let $K$ to be the set of all possible $k$. Similarly, we can define $k_{-i}$ and $K_{-i}$, $i = 1, 2, ..., M$.

Given action profile $a = \{a_1, a_2, ..., a_M\}$ of the $M$ players, the payoff of player $i$ is defined as

$$U_i(a) = q_i \left\{ \sum_{k_{-i} \in K_{-i}} \left( \prod_{j \neq i} q_j^{k_j} (1 - q_j)^{1-k_j} \right) \Pr(i|k_{-i}, x) \right\} v - x_i^0 - \Delta, i = 1, 2, ..., M,$$

where $\Pr(i|k_{-i}, x) = \frac{x_i^r}{\sum_j x_j^r + \sum_{j \neq i} k_j x_j^r}$ if $x_i^r + \sum_{j \neq i} k_j x_j^r > 0$, and $\Pr(i|k_{-i}, x) = \frac{1}{\sum_{j \neq i} k_j}$ if $x_i^r + \sum_{j \neq i} k_j x_j^r = 0$. Note that $\Pr(i|k_{-i}, x)$ equals to the winning probability of an entrant $i$ when the entry status of others is denoted by $k_{-i}$ and players’ effort is $x$ if they enter.

Note that this game is a symmetric game as defined by Dasgupta and Maskin (1986) in their Definition 7. We will apply their Theorem 6* in Appendix to establish the existence of symmetric equilibrium in mixed strategy.

In what follows, we show that for each $i$, the discontinuities of $U_i$ are confined to a subset of a continuous manifold of dimension less than $M$ as required by page 7 of Dasgupta and Maskin (1986). Following the notations on page 22 of Dasgupta and Maskin (1986). Let $Q = \{2\}$, $D(i) = 1$, and $f_{ij}^1$ to be an identity function. Following their (A1) of page 22, we define manifold $A^*(i) = \{a \in A| \exists j \neq i, \exists k \in Q, \exists d, 1 \leq d \leq D(i) \text{ such that } a_{jk} = f_{ij}^d(a_{ik})\}$. Clearly, $A^*(i)$ is of dimension less than $M$. The set of discontinuous points for $U_i(a)$ can be written as $A^{**}(i) = \{a \in A|q_j x_j = 0, \forall j = 1, 2, ..., M; q_i > 0, x_i = 0; \exists j_0 \neq i, \text{ such that } q_{j_0} > 0 \text{ and } x_{j_0} = 0\}$. Clearly, $A^{**}(i) \subset A^*(i)$, since any element in $A^{**}(i)$ must satisfy the following conditions: For $k = 2 \in Q, \exists j_0 \neq i$, such that $x_{j_0} = f_{ij}^1(x_{ik})$, i.e. $a_{j_0i} = f_{ij}^1(a_{i2})$. According to their Theorem 6*, we need to verify the following conditions hold.
First, as constructed above, \( U_i(a) \) is continuous except on a subset \( A^\star(i) \) of \( A^*(i) \), where \( A^*(i) \) is defined by (A1).

Second, clearly, we have \( \sum_i U_i(a) = \nu[1 - \prod_i (1 - q_i)] - \sum_i q_i(x_i^\alpha + \Delta) \), which is continuous and thus upper semi-continuous.

Third, \( U_i(a) \) clearly is bounded on \( A = [0, 1] \times [0, v^{1/\alpha}] \).

Fourth, we verify that Property \((\alpha^*)\) of page 24 is satisfied. Define \( B^2 \) as the unit circle with the origin as its center, i.e. \( B^2 = \{e = (q, x) \mid q^2 + x^2 = 1\} \). Pick up any continuous density function \( v(\cdot) \) on \( B^2 \) such that \( v(e) = 0 \) iff \( e_1 \leq 0 \) or \( e_2 \leq 0 \). Note that \( U_i(a_i, a_{-i}) \) is continuous in \( a_{i1} \) and lower semi-continuous in \( a_{i2} \). \( \forall a = (\bar{a}_i, a_{-i}) \in A^\star(i) \), clearly we have that for any \( e \) such that \( v(e) > 0 \) (i.e. \( \min(e_1, e_2) > 0 \)), \( \liminf_{\theta \to 0^+} U_i(\bar{a}_i + \theta e, a_{-i}) > U_i(\bar{a}_i, a_{-i}) \) as \( \theta \to 0 \), \( e_2 > 0 \) and \( q_i > 0 \), \( x_i = 0 \) in \( \bar{a}_i \). This leads to that \( \int_{B^2} \liminf_{\theta \to 0^+} U_i(\bar{a}_i + \theta e, a_{-i})v(e)de > U_i(\bar{a}_i, a_{-i}), \forall \bar{a}_i \in A^*_{i1}(i), a_{-i} \in A^*_{i1}(\bar{a}_i), \) where \( A^*_{i1}(i) \) is the collection of all \( \bar{a}_i \) of player \( i \) that appear in \( A^\star(i) \), \( A^*_{i1}(\bar{a}_i) \) is the collection of others’ actions \( a_{-i} \) such that \( a = (\bar{a}_i, a_{-i}) \in A^\star(i) \). This confirms that Property \((\alpha^*)\) holds for the above game.

Thus according to Theorem 6* of Dasgupta and Maskin (1986), there exists a symmetric mixed strategy equilibrium. Without loss of generality, we use \( \mu_i(q) \) to denote the equilibrium probability measure of action \( q \), and use \( \mu_i(x) \) to denote the equilibrium probability measure of action \( x \).

Next we show that for any strategy profile of players \( \{(\mu_{i1}(q_i), \mu_{i2}(x_i))\} \). The players’ payoffs are same from strategy profile of players that is defined as \( \{(E_{\mu_{i1}}q_i, \mu_{i2}(x_i))\} \). The expected utility of player \( i \) from profile \( \{(\mu_{i1}(q_i), \mu_{i2}(x_i))\} \) is

\[
E_aU_i(a) = E_q\{E_{q_{-i}}E_x[q_i \sum_{k_{-i},i \in K_{-i}, j \neq i} \left( \prod_{j \neq i} q_j^{k_j} (1 - q_j)^{1-k_j} \right) \Pr(i|k_{-i}, x)]v - x_i^\alpha - \Delta]\}
\]

\[
= E_q\{q_iE_x\sum_{k_{-i},i \in K_{-i}, j \neq i} \left( \prod_{j \neq i} q_j^{k_j} (1 - q_j)^{1-k_j} \right) \Pr(i|k_{-i}, x)]v - x_i^\alpha - \Delta]\}
\]

\[
= E_{q_i}E_x\left[ \sum_{k_{-i},i \in K_{-i}, j \neq i} \left( \prod_{j \neq i} (E_{q_j})^{k_j} (1 - E_{q_j})^{1-k_j} \right) \Pr(i|k_{-i}, x)]v - x_i^\alpha - \Delta\right], \forall i. \quad (16)
\]

The above result means that given others take strategy \( (E_{\mu_i}q_i, \mu_{i2}(x_i)) \), the same strategy is also the best strategy for player \( i \). Otherwise, \( (\mu_1(q), \mu_2(x)) \) would not be the optimal strategy for player \( i \) when others take the same strategy \( (\mu_1(q), \mu_2(x)) \). Therefore, \( (E_{\mu_i}q_i, \mu_{i2}(x)) \) is a symmetric equilibrium for the above game.

It is easy to see that \( (q^*, \mu^*(x)) = (E_{\mu_i}q_i, \mu_{i2}(x)) \) is a symmetric equilibrium for our original game based on the way the extended game is constructed. \( U_i(a) \) equals player \( i \)'s expected payoffs when he enters with probability \( q_i \) and exerts effort \( x_i \) when he enters, given that other bidder \( j \) enters with probability \( q_j \) and exerts effort \( x_j \) when he enters. This claim also holds when they adopt any other entry strategies with measure \{\( \mu_{i1}(q), i = 1, 2, ..., M \)\} due to (16).
According to (16), only the expected entry probabilities \( \{E_{\mu_i}q, i = 1, 2, ..., M\} \) count.

Note we must have \( q^* = E_{\mu_i}q \in (0, 1) \). First, \( q^* = E_{\mu_i}q = 0 \) cannot be an entry equilibrium when \( \Delta < v \) (Assumption 1). Second, \( q^* = E_{\mu_i}q = 1 \) cannot be an entry equilibrium when \( \Delta > \frac{v}{M} \) (Assumption 1). The expected equilibrium payoff of players must be nonnegative. Thus we must have
\[
\left(1 - \frac{(1 - E_{\mu_i}q)^M}{M}v - M(E_{\mu_i}q)[\Delta + E_{\mu_i}x]\right) \geq 0.
\]
This leads to \( (1 - (1 - E_{\mu_i}q)^M)v - M(E_{\mu_i}q)\Delta > 0 \). Thus \( q^* = E_{\mu_i}q < \tilde{q} \) by Definition 1 and proof of Lemma 1.

**Part (b):** The equilibrium payoff cannot be negative. When \( q^* = E_{\mu_i}q \in (0, 1) \), we must have the equilibrium payoffs of player to be zero as otherwise it cannot be an equilibrium as the player would enter with probability 1 and earn a positive payoff. ■

**Proof of Lemma 2**

**Proof.** If a symmetric equilibrium with pure strategy bidding exists, according to the first order condition \( \frac{d\pi_i(x_i)}{dx_i} = 0 \) and the symmetry condition \( x_i = x \), \( x^* \) must solve
\[
\sum_{N=1}^{M} C_{M-1}^{N-1}q^{N-1}(1 - q)^{M-N} \frac{(N - 1)rv}{N^2x^*} - \alpha x^{*\alpha - 1} = 0,
\]
which yields
\[
x^*(q) = \left[ \sum_{N=1}^{M} C_{M-1}^{N-1}q^{N-1}(1 - q)^{M-N} \frac{N - 1 rv}{N^2 \alpha} \right]^{\frac{1}{\alpha}}.
\]

The equilibrium expected payoff is
\[
\pi^*(x^*(q), q) = \sum_{N=1}^{M} C_{M-1}^{N-1}q^{N-1}(1 - q)^{M-N} \frac{v}{N} x^*(q)
\]
\[-\left[ \sum_{N=1}^{M} C_{M-1}^{N-1}q^{N-1}(1 - q)^{M-N} \frac{N - 1 rv}{N^2 \alpha} \right] x^*(q)
\]
\[= \sum_{N=1}^{M} C_{M-1}^{N-1}q^{N-1}(1 - q)^{M-N} \frac{v}{N} \left(1 - \frac{N - 1 rv_1}{N_1} \right).\]

By entering the contest and submit the bid \( x^*(q) \), every potential contestant \( i \) ends up with an expected payoff
\[
\pi^*(x^*(q), q) - \Delta.
\]

By Theorem 1 (b), each potential bidder receives a zero expected payoff for the equilibrium entry \( q^* \), i.e. \( \pi^*(x^*(q^*), q^*) = \Delta \).

The expected overall effort of the contest \( (x_T^*) \) obtains as
\[
x_T^* = M q^* x^*(q^*)
\]
\[= M q^* \left[ \sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1 - q^*)^{M-N} \frac{N - 1 rv_1}{N_1} \alpha \right]^{\frac{1}{\alpha}}.
\]
Proof of Lemma 3

Proof. By Lemma 2, \( q^* \) satisfies

\[
F(q^*, r) = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1} (1 - q^*)^{M-N} \frac{v}{N} (1 - \frac{N-1}{N} \frac{r}{\alpha}) - \Delta = 0.
\]

Apparently, \( F(q^*, r) \) is continuous in and differentiable with both arguments. We first claim that \( F(q^*, r) \) strictly decreases with \( q^* \). Define \( \pi_N = \frac{v}{N} (1 - \frac{N-1}{N} \frac{r}{\alpha}) \). Taking its first order derivative yields

\[
\frac{F(q^*, r)}{dq^*} = \sum_{N=1}^{M} C_{M-1}^{N-1} [(N-1)q^{N-2}(1 - q^*)^{M-N} - (M-N)q^{N-1}(1 - q^*)^{M-N-1}] \pi_N
\]

\[
= \sum_{N=1}^{M} C_{M-1}^{N-1}(N - 1)q^{N-2}(1 - q^*)^{M-N} \pi_N - \sum_{N=1}^{M} C_{M-1}^{N-1}(M - N)q^{N-1}(1 - q^*)^{M-N-1} \pi_N
\]

\[
= (M - 1)\left\{ \sum_{N=2}^{M} C_{M-2}^{N-2}q^{N-2}(1 - q^*)^{M-N} \pi_N - \sum_{N=1}^{M} C_{M-1}^{N-1}q^{N-1}(1 - q^*)^{M-N-1} \pi_N \right\}
\]

\[
= (M - 1) \sum_{N=1}^{M-1} C_{M-2}^{N-1}q^{N-1}(1 - q^*)^{M-N-1} (\pi_{N+1} - \pi_N),
\]

which is obviously negative because \( \pi_N = \frac{v}{N} \left[ 1 - \left( 1 - \frac{1}{N} \frac{r}{\alpha} \right) \right] v \geq 0 \) and it monotonically decreases with \( N \).

When all other potential contestants play \( q = 0 \), a potential contestant receives a payoff \( v - \Delta > 0 \), and he must enter with probability one. When all others play \( q = \bar{q} \), a participating contestant receives negative expected payoff if he enters by Definition 1 and Lemma 1 ((1 – \( \bar{q} \))^{M-1}v \(< \Delta \)), which cannot constitute an equilibrium either. Hence, a unique \( q^* \in (0, \bar{q}) \) must exist that solves \( \pi^*(x^*, q) = \Delta \). Each potential contestant is indifferent between entering and staying inactive when all others play the strategy. This constitutes an equilibrium.

Moreover, \( F(q^*, r) \) strictly decreases with \( r \). Since it also strictly decreases with \( q^* \), the part (b) of the lemma is then verified. ■

Proof of Lemma 4

Proof. Denote \( k_i = x_i^t \), \( k^* = x^{s\alpha} \), \( t = \frac{r}{\alpha} \in (0, \frac{M-1}{M-2}] \), then \( \tilde{\pi}_i(x_i) \) can be rewritten as

\[
\tilde{\pi}_i(k_i) = \sum_{N=1}^{M} C_{M-1}^{N-1}q^{N-1}(1 - q^*)^{M-N} \frac{k_i^t}{k_i^t + (N-1)k^{s\alpha}v} - k_i,
\]

Evaluating \( \tilde{\pi}_i \) with respect to \( k_i \) yields

\[
\frac{d\tilde{\pi}_i}{dk_i} = \sum_{N=1}^{M} C_{M-1}^{N-1}q^{N-1}(1 - q^*)^{M-N} \frac{(N-1)tk_i^{t-1}k^{s\alpha}v}{[k_i^t + (N-1)k^{s\alpha}]^2} - 1.
\]
Note

\[ k^* = \sum_{N=1}^{M} C_{M-1}^N q^{N-1}(1 - q^*)^{M-N} \frac{N-1}{N^2} tN. \]

To verify that \( k^* \) is the global maximizer of \( \bar{\pi}_i(k_i) \) given that all other participants exert the same effort. Define \( p_t(k_i, k_{-i}; N) = \frac{k_i^t}{k_i^t + (N-1)k_i} \). One can verify \( \xi_N(\bar{k}_i) = \left. \frac{\partial^2 p_t(k_i, k_{-i}; N)}{\partial k_i^2} \right|_{k_{-i}=k^*} = \frac{-(t+1)k_i^t + (t-1)(N-1)k^*}{[k_i^t + (N-1)k^*]t} t(k_i^t - 2(N-1)k^*). \) It implies that \( \Phi_N(\bar{k}_i) = \left. \frac{\partial p_t(k_i, k_{-i}; N)}{\partial k_i} \right|_{k_{-i}=k^*} \) is not monotonic:

It is positive if \( k_i^t < \frac{t-1}{t+1}(N-1)k^* \), and negative if \( k_i^t > \frac{t-1}{t+1}(N-1)k^* \). Clearly \( \frac{t-1}{t+1}(N-1) \leq 1 \) if and only if \( t \leq \frac{N}{N-2} \). Because \( t \leq 1 + \frac{1}{M-2} \), we must have \( \frac{t-1}{t+1}(N-1) < 1 \) for all \( N \leq M \).

Let \( \Phi(k_i) = \sum_{N=1}^{M} C_{M-1}^N q^{N-1}(1 - q^*)^{M-N} \frac{\partial p_t(k_i, k_{-i}; N)}{\partial k_i} \big|_{k_{-i}=k^*} \), and \( \xi(k_i) = \sum_{N=1}^{M} C_{M-1}^N q^{N-1}(1 - q^*)^{M-N} \frac{\partial^2 p_t(k_i, k_{-i}; N)}{\partial k_i^2} \big|_{k_{-i}=k^*} \). The above results imply that \( k_i^t > \frac{t-1}{t+1}(N-1)k^* \) when \( k_i = k^* \) for all \( N \leq M \), which means that \( \xi(k_i)|_{k_{-i}=k^*} < 0 \). This leads to that \( \frac{\partial^3 \bar{\pi}_i(k_i)}{\partial k_i^3} \big|_{k_{-i}=k^*} = v \bar{\pi}_i(k_i)|_{k_{-i}=k^*} < 0 \). Hence, \( k_i = k^* \) must be at least a local maximizer of when \( k_{-i} = k^* \).

Since when \( k_i < \left[ \frac{t-1}{t+1} \right]^{1/t} k^* \), \( \xi_N(\bar{k}_i) > 0 \) for all \( N \leq M \), we have \( \xi(k_i) > 0 \) when \( k_i < \left[ \frac{t-1}{t+1} \right]^{1/t} k^* \), which means that \( \Phi(k_i) \) increases when \( k_i < \left[ \frac{t-1}{t+1} \right]^{1/t} k^* \). Similarly, \( \xi(k_i) < 0 \) when \( k_i > \left[ \frac{t-1}{t+1}(M-1) \right]^{1/t} k^* \), which means that \( \Phi(k_i) \) decreases when \( k_i > \left[ \frac{t-1}{t+1}(M-1) \right]^{1/t} k^* \). We next show that there exists a unique \( k' \in (\left[ \frac{t-1}{t+1} \right]^{1/t} k^* , \left[ \frac{t-1}{t+1}(M-1) \right]^{1/t} k^*) \) such that \( \Phi(k_i) \) increases (decreases) if and only if \( k_i < (>) k' \). For this purpose, it suffices to show that there exists a unique \( k' \in (\left[ \frac{t-1}{t+1} \right]^{1/t} k^* , \left[ \frac{t-1}{t+1}(M-1) \right]^{1/t} k^*) \), such that \( \xi(k_i') = 0 \).

First, such \( k' \) must exist by continuity of \( \xi(k_i) \). As have been revealed, \( \xi(k_i) > 0 \) when \( k_i < \left[ \frac{t-1}{t+1} \right]^{1/t} k^* \); and \( \xi(k_i) < 0 \) when \( k_i > \left[ \frac{t-1}{t+1}(M-1) \right]^{1/t} k^* \).

Second, the uniqueness of \( k' \) can be verified as below. We have

\[
\frac{\partial^3 p_t(k_i, k_{-i}; N)}{\partial k_i^3} \bigg|_{k_{-i}=k^*} = t(N-1)k^* \left\{ \begin{array}{l}
(t-2) \frac{k_i^t - 3 - (t+1)k_i^t + (t-1)(N-1)k^*}{[k_i^t + (N-1)k^*]^3} \\
+ k_i^t - 3 \left[ t(t+1)k_i^t + (t-1)(N-1)k^* \right]^2 \\
+ \frac{2(k_i^t + (N-1)k^*)}{[(t-1)k_i^t + (t-1)(N-1)k^*]^2} (t+1)k_i^t - 2(t-1)(N-1)k^* \end{array} \right. \]

\[
= t(N-1)k^* \left\{ \begin{array}{l}
(t-2) \frac{[-(t+1)k_i^t + (t-1)(N-1)k^*]}{[k_i^t + (N-1)k^*]^3} \\
+ \frac{2(k_i^t + (N-1)k^*)}{[(t-1)k_i^t + (t-1)(N-1)k^*]^2} \end{array} \right. \]
Recall $\xi_N(k_i) = \frac{-(t+1)k_i^t + (t-1)(N-1)k^t}{[k_i^t + (N-1)k^t]^4} t k_i^{-2}(N-1)k^t$. We then have
\[
\frac{\partial^3 p_i(k_i, k_{-i}; N)}{\partial k_i^3} \bigg|_{k_{-i}=k^*} = (t - 2)k_i^{-1} \xi_N(k_i) + 2t^2(N - 1)k^t k_i^{2t-3} [\frac{1}{k_i^t + (N-1)k^t}]^4 \bigg[ (t + 1)k_i^t - (2t - 1)(N - 1)k^t \bigg].
\]

We now claim $[(t + 1)k_i^t - (2t - 1)(N - 1)k^t]$ is negative for all $k_i \leq \frac{[t+1](M - 1)}{t+1}$. A detailed proof is as follows. From $k_i \leq \frac{[t+1](M - 1)}{t+1}$, we have $(t + 1)k_i^t \leq (t - 1)(M - 1)k^t$. To show $(t + 1)k_i^t - (2t - 1)(N - 1)k^t < 0$, it suffices to show $(t - 1)(M - 1) < (2t - 1)(N - 1)$ when $N = 2$, which requires $t < 1 + \frac{1}{M-2}$. This holds as $t \leq 1 + \frac{1}{M-2}$.

We thus have at any $k_i \in \big(\frac{[t+1](M - 1)}{t+1}, \frac{[t+1](M - 1)}{t+1}\big)$ such that $\xi(k_i) = 0$, $\xi(k_i)$ must be locally decreasing, because
\[
\frac{\partial \xi(k_i)}{\partial k_i} = (t - 2)k_i^{-1} \sum_{N=1}^{M} C_N \tbinom{N}{M} q^{N-1}(1 - q^*) M - N \xi_N(k_i) + \sum_{N=1}^{M} C_{M-1} q^{N-1}(1 - q^*) M - N A_N(k_i) = \sum_{N=1}^{M} C_{M-1} q^{N-1}(1 - q^*) M - N A_N(k_i) < 0
\]
as $A_N(k_i) = \frac{2t^2(N - 1)k^t k_i^{2t-3}}{[k_i^t + (N-1)k^t]^4} \bigg[ (t + 1)k_i^t - (2t - 1)(N - 1)k^t \bigg] < 0$.

We are ready to show the uniqueness of $k'$ by contradiction. Suppose that there exists more than one zero points $k'$ and $k''$ with $k' \neq k''$ for $\xi(k_i)$. Because $\xi(k_i)$ must be locally decreasing, then there must exist at least another zero point $k''' \in (k', k'')$ at which $\xi(k_i)$ is locally increasing. Contradiction thus results. Hence, such a zero point $k'$ of $\xi(k_i)$ must be unique.

Recall $\Phi(k_i)$ increases (decreases) if and only if $k_i < (>) k'$ and it reaches its maximum at $k'$. Note $\frac{\partial \tilde{\pi}_i(k_i)}{\partial k_i} = v\Phi(k_i) - 1$ and $\Phi(0) = 0$. Therefore $\frac{\partial \tilde{\pi}_i(k_i)}{\partial k_i} |_{k_i=0} < 0$. Thus $\frac{\partial \tilde{\pi}_i(k_i)}{\partial k_i}$ has exactly two zero points with the smaller one ($k_s$) being the local minimum point of $\tilde{\pi}_i(k_i)$. Note $k_i = k^*$ must be a zero point for $\frac{\partial \tilde{\pi}_i(k_i)}{\partial k_i}$ by definition. Since $k_i = k^*$ is a local maximum point of $\tilde{\pi}_i(k_i)$, it is higher than other zero point ($k_s$) of $\frac{\partial \tilde{\pi}_i(k_i)}{\partial k_i}$ which is a local minimum point of $\tilde{\pi}_i(k_i)$.

Note $x_m = (k_s)^{1/\alpha}$ is the unique local minimum of $\tilde{\pi}_i(x_i)$, and note $x^* = (k_i)^{1/\alpha}$ is the unique inner local maximum of $\tilde{\pi}_i(x_i)$. Note $x_m < x^*$. The results of Lemma 4 are shown.

**Proof of Lemma 5**

**Proof.** Define an increasing transformation of $\bar{x}_T(q)$:
\[
\Psi(q) = [\bar{x}_T(q)]^\alpha = (Mq)^{\alpha-1} \left\{ [1 - (1-q)^M]v - Mq\Delta \right\}
\]

Note that $\Psi(q)|_{q=0} = 0$; and $\Psi(q)|_{q=1} = M^{\alpha-1}(v - M\Delta) < 0$ since $\frac{v}{M} < \Delta$ (Assumption
Since we have a pure-strategy bidding, an overall effort of both is negative when there exists a unique \((0 < q < 1)\) such that \(f(q) = 0\). Proof of Lemma 3 has shown that \(f(q)\) decreases with both \(q\) and \(r\). Thus the zero point \((\hat{q})\) of \(f(q)\) must fall in \([q_0, \tilde{q}]\).

**Proof of Theorem 4**

**Proof.** Proof of Lemma 3 has shown that \(F(q, r) = \sum_{N=1}^{M} C_{M-1}^{N-1} q^{N-1}(1-q)^{M-N} v \left(1 - \frac{N-1}{N} \frac{r}{\alpha} \right)^{-\Delta}\) decreases with both \(q\) and \(r\). Thus \(F(q, r) = 0\) uniquely defines \(r\) as a decreasing function of \(q\). Since \(F(q_0, r_0) = 0\) and \(\hat{q} > q_0\), we must have \(r(\hat{q}) < r_0\). Theorem 3 thus means that contest \(r(\hat{q})\) would induce entry equilibrium \(\hat{q}\) and pure-strategy bidding whenever \(r(\hat{q}) \leq \alpha(1 + \frac{1}{M-2})\). Since we have a pure-strategy bidding, an overall effort of \(\bar{\pi}(\hat{q})\) clearly is induced at the equilibrium.

Consider any other \(r \neq r(\hat{q})\). If \(r\) induces equilibrium entry \(q(r)\) and pure-strategy bidding, then the total effort induced is \(\bar{\pi}(q(r))\). Note that by Lemma 3, equilibrium \(q(r)\) decreases with \(r\). Thus \(r \neq r(\hat{q})\) means \(q(r) \neq \hat{q}\). \(\bar{\pi}(q)\) is single peaked at \(q\) according to Lemma 5. Thus for any \(r \neq r(\hat{q})\), we must have \(\bar{\pi}(q(r)) < \bar{\pi}(\hat{q})\). If \(r\) induces equilibrium entry \(q(r)\) and mixed-strategy bidding, then the total expected effort induced is strictly lower than \(\bar{\pi}(\hat{q})\) when \(\alpha > 1\), based on the arguments deriving this boundary in Section 4.1. Therefore the total effort induced must be strictly lower than \(\bar{\pi}(\hat{q})\).  ■
Proof of Lemma 6

Proof. By definition \( \frac{\partial \mathcal{T}}{\partial M'}(M') = \frac{\partial \mathcal{T}}{\partial M'}(\hat{q}(M'); M'). \)

By Envelope Theorem, \( \frac{\partial \mathcal{T}(\hat{q}(M'); M')}{\partial M'} = \frac{\partial \mathcal{T}(q; M')}{\partial M'}|_{q=\hat{q}(M')} \). We have

\[
\frac{\partial \mathcal{T}(q; M')}{\partial M'}|_{q=\hat{q}(M')} = \partial \left[ (M' \hat{q}(M'))^{\frac{\alpha-1}{\alpha}} \left\{ [1 - (1 - \hat{q}(M'))^M]v - M' \hat{q}(M') \Delta \right\} \right] / \partial M' \\
= \frac{\alpha - 1}{\alpha} M' \hat{q}(M') \left\{ [1 - (1 - \hat{q}(M'))^M]v - M' \hat{q}(M') \Delta \right\}^{\frac{1}{\alpha}} \\
+ \frac{1}{\alpha} (\hat{q}(M'))^{\frac{\alpha-1}{\alpha}} \left\{ [1 - (1 - \hat{q}(M'))^M]v - M' \hat{q}(M') \Delta \right\}^{\frac{1}{\alpha} - 1} \\
\times [- (1 - \hat{q}(M'))^M v \ln(1 - \hat{q}(M')) - \hat{q}(M') \Delta],
\]
which has the same sign as

\[
\lambda = (\alpha - 1) \left\{ [1 - (1 - \hat{q}(M'))^M]v - M' \hat{q}(M') \Delta \right\} + M' \left\{ - (1 - \hat{q}(M'))^M v \ln(1 - \hat{q}(M')) - \hat{q}(M') \Delta \right\}.
\]

Because \(- \ln(1 - \hat{q}(M')) < \frac{\hat{q}(M')}{1 - \hat{q}(M')}, \)
we have \( M' \left\{ - (1 - \hat{q}(M'))^M v \ln(1 - \hat{q}(M')) - \hat{q}(M') \Delta \right\} < \hat{q}(M') \left\{ M'(1 - \hat{q}(M'))^M v - M' \Delta \right\}. \)
Hence, \( \lambda < (\alpha - 1) \left\{ [1 - (1 - \hat{q}(M'))^M]v - M' \hat{q}(M') \Delta \right\} + \hat{q}(M') \left\{ M'(1 - \hat{q}(M'))^M v - M' \Delta \right\} = 0 \) (by the definition of \( \hat{q}(M') \)). We then have \( \frac{\partial \mathcal{T}(\hat{q}(M'); M')}{\partial M'} < 0. \)

Proof of Theorem 6

Proof. We first show the following claim for a subgame with \( N \) players.

Claim: For \( N \leq M \) such that \( \frac{N}{N-1} < \frac{\alpha}{2} \), there exists a symmetric mixed strategy equilibrium for the \( N \)-player subgame. The equilibrium payoff of a player \( \pi^d_N \) falls in \([0, \frac{N}{2}]\).

The proof of this claim relies on Theorem 6 of Dasgupta and Maskin (1986). The application of their Theorem 6 requires four conditions as has been pointed out by Baye et al (1994) who have shown the existence of a symmetric mixed-strategy equilibrium when \( N = 2 \) and effort costs are linear. However, when effort costs are nonlinear and \( N > 2 \), the proof is almost identical. Condition (i) requires that the discontinuity set \( S_i \) of player \( i \)'s payoff is confined to a subset of a continuous manifold of dimension less than \( N \). Let this manifold be defined as \( A^*(i) = \{ x | x_1 = x_2 = \ldots = x_N \} \), which has a zero measure. The only discontinuity point of player \( i \)'s payoff is \((0, 0, \ldots, 0) \in A^*(i) \). Thus condition (i) holds. Condition (ii) of this theorem requires that the sum of players' payoffs must be upper semi-continuous. From (2), we have that this sum is \( v - \sum_i x_i^2 \), which is continuous and therefore upper semi-continuous. Condition (iii) requires that player \( i \)'s payoff is bounded. This clearly holds as it falls in \([-v, v] \).
when \( x_i \in [0, v^{1/\alpha}] \). Note that a player never bids higher than \( v^{1/\alpha} \). Condition (iv) requires that player \( i \)'s payoff must be weakly lower semi-continuous. The only point one needs to check is the discontinuity point \((0, 0, \ldots, 0)\). At this point, player \( i \)'s payoff is lower semi-continuous, and thus is weakly lower semi-continuous. Since all four conditions required are satisfied. The existence of a symmetric mixed-strategy equilibrium is guaranteed by Theorem 6 in Dasgupta and Maskin (1986).

In a symmetric equilibrium, every contestant wins the prize \( v \) with the same probability, and they incur positive effort costs.\(^{15}\) Therefore, the equilibrium payoff must be lower than \( \frac{v}{N} \).

We now introduce the definition of a symmetric entry equilibrium. Entry probability \( q_d^* \in [0, 1] \) constitutes a symmetric entry equilibrium if and only if

\[
\sum_{N=1}^{M} C_{M-1}^{N-1} q_d^{N-1}(1 - q_d^*)^{M-N} \pi_N^d = \Delta, \text{ if } q_d^* \in (0, 1),
\]

\[
\pi_M^d \geq \Delta, \text{ if } q_d^* = 1,
\]

\[
\pi_1^d = v < \Delta, \text{ if } q_d^* = 0.
\]

We now are ready to show a symmetric entry equilibrium exists which must fall into \((0, 1)\).

Note that with Assumption 1, both \( q_d^* = 1 \) and \( q_d^* = 0 \) cannot be an entry equilibrium. The existence of symmetric entry equilibria depends on the existence of the solution of \( \sum_{N=1}^{M} C_{M-1}^{N-1} q_d^{N-1}(1 - q_d^*)^{M-N} \pi_N^d = \Delta \). Note the left hand side is continuous in \( q_d^* \). When \( q_d^* = 0 \), it is lower than the right hand side. When \( q_d^* = 1 \), it is higher than the right hand side. Therefore, there must exist \( q_d^* \in (0, 1) \) such that \( \sum_{N=1}^{M} C_{M-1}^{N-1} q_d^{N-1}(1 - q_d^*)^{M-N} \pi_N^d = \Delta \).

\[\blacksquare\]

**Proof of Lemma 7**

**Proof.** Under policy \( d \), for a given \( r \in (0, \alpha(1 + \frac{1}{M-1})) \) the subgame boils down to a standard symmetric \( N \)-player contest. Whenever \( N \geq 2 \), each representative participant \( i \) chooses his bid \( x_i \) to maximize his expected payoff

\[
\pi_i = p_N(x_i, x_{-i}) v - x_i^0,
\]

where \( p_N(x_i, x_{-i}) \) is given by the contest success function (1). Standard technique leads to the well known results in contest literature. In the unique symmetric pure-strategy Nash equilibrium, each participant bids

\[
x_N = \left( \frac{N - 1}{N^2} \frac{r v}{\alpha} \right)^{\frac{1}{\alpha}}.
\]

\(^{15}\)Clearly, exerting a zero effort is not an equilibrium.
Each participating contestant earns an expected payoff

$$\pi_N = \frac{v}{N} \left(1 - \frac{N - 1}{N} \frac{r}{\alpha}\right).$$

Note that $x_N$ reduces to zero, and $\pi_N$ amounts to $v$ if $N = 1$, i.e. nobody else enters the contest. Suppose that all others choose a strategy $q_d \in [0, 1]$. A potential contestant $i$ ends up with an expected payoff

$$u_i(q) = \sum_{N=1}^{M} C_{M-1}^{N-1} q_d^{N-1} (1 - q_d)^{M-N} \pi_N - \Delta.$$

By proof of Lemma 3, $\pi(q_d)$ strictly decreases with $q_d$. There must exist a unique $q^*_d \in (0, 1)$ that solves $\pi^*_d = \pi^*_d(q_d) = \Delta$. Each potential contestant is indifferent between entering and staying inactive when all others play the strategy. This constitutes an equilibrium.

Since each $N-$ player contest elicits a total bid $N \cdot x_N \equiv N \left(\frac{N - 1}{N^2} \frac{r}{\alpha}\right)$. Hence, expected overall bid is obtained as

$$x^*_T (r, d) = \sum_{N=1}^{M} C_{M-1}^{N-1} q^*_d^{N-1} (1 - q^*_d)^{M-N} \left(\frac{N - 1}{N^2} \frac{r}{\alpha}\right)^{\frac{1}{\alpha}} = M q^*_d \sum_{N=1}^{M} C_{M-1}^{N-1} q^*_d^{N-1} (1 - q^*_d)^{M-N} \left(\frac{N - 1}{N^2} \frac{r}{\alpha}\right)^{\frac{1}{\alpha}}.$$

**Proof of Theorem 7**

**Proof.** For a given $r$, concealment and disclosure yields the same equilibrium entry strategy, i.e., $q^*_c = q^*$. Potential contestants are ex ante indifferent between concealment and disclosure. This claim can be directly verified by the proofs of Lemmas 2 and 7. $q^*$ and $q^*_d$ solve the same equations (4) and (14). By Jensen’s inequality, $\frac{1}{\alpha} \leq 1$ implies that $x^*_T (r, c) \geq x^*_d(r, d)$ because

$$\left(\sum_{N=1}^{M} C_{M-1}^{N-1} q^*_c^{N-1} (1 - q^*_c)^{M-N} \frac{r}{\alpha}\right)^{\frac{1}{\alpha}} \geq \sum_{N=1}^{M} C_{M-1}^{N-1} q^*_d^{N-1} (1 - q^*_d)^{M-N} \left(\frac{N - 1}{N^2} \frac{r}{\alpha}\right)^{\frac{1}{\alpha}}.$$

**Proof of Remark 1**

**Proof.** Fix $\alpha = 1$. When $r \in (\alpha(1 + \frac{1}{M-1}), \alpha(1 + \frac{1}{M-2})]$, the symmetric equilibrium probability $q_c \in (0, 1)$ of a contest under policy $c$ is determined by the break-even condition

$$v \sum_{N=1}^{M} C_{M-1}^{N-1} q_c (1 - q_c)^{M-N} \left[1 - \frac{N - 1}{N^2} \frac{r}{\alpha}\right] = \Delta.$$

However, under policy $d$, the break-even condition that determines equilibrium entry probability is

$$v \sum_{N=1}^{M} C_{M-1}^{N-1} q_d (1 - q_d)^{M-N} \pi_N = \Delta.$$
where $\pi_N$ is the equilibrium payoff of a participating bidder in a subgame with a total of $N$ participants. For small $N$ such that $r \leq \alpha(1 + \frac{1}{N-1})$, $\pi_N = \frac{1}{N} - \frac{N-1}{N^2} \frac{r}{\alpha} \geq 0$. However, for sufficiently large $N$ such that $\frac{1}{N} - \frac{N-1}{N^2} \frac{r}{\alpha} < 0$, $\pi_N$ must be strictly greater than $\frac{1}{N} - \frac{N-1}{N^2} \frac{r}{\alpha}$, because it must be nonnegative by individual rationality. Hence, we must have

$$v \sum_{N=1}^{M} C_{M-1}^{N-1} q(1-q)^{M-N} \pi_N > v \sum_{N=1}^{M} C_{M-1}^{N-1} q(1-q)^{M-N} \left[ \frac{1}{N} - \frac{N-1}{N^2} \frac{r}{\alpha} \right]$$

for $q \in (0,1)$. This implies generally $q_d \neq q_c$ for the given $r$, which further means that the total effort induced would generally be different. Note for $\alpha = 1$, the total effort induced is completely determined by the entry probability.

**Proof of Theorem 8**

**Proof.** First note that at any symmetric equilibrium when the number of bidders is disclosed, every bidder enjoys zero payoff. Therefore, we have $[1 - (1 - q_d^*)^M]v = Mq_d^* \{\Delta + E_N[(x_N)^{\alpha}]\}$, i.e. $E_NE[(x_N)^{\alpha}] = [Mq_d^*]^{-1} [1 - (1 - q_d^*)^M]v - \Delta$, where $x_N$ denotes the equilibrium individual effort in a subgame with $N$ contestants. The expected total effort at the equilibrium is $Mq_d^* E_N[E(x_N)] = Mq_d^* E_NE\{[(x_N)^{\alpha}]^{1/\alpha}\} \leq Mq_d^* E_N\{E[(x_N)^{\alpha}]\}^{1/\alpha} \leq Mq_d^* \{E_NE[(x_N)^{\alpha}]\}^{1/\alpha} = [Mq_d^*]^{\frac{\alpha-1}{\alpha}} \times \{[1 - (1 - q_d^*)^M]v - Mq_d^*\Delta\}^{\frac{1}{\alpha}}$ as $\alpha \geq 1$. Note that the last expression is identical to the right hand side of (11). When $r(\hat{q})$ induces entry $\hat{q}$ and pure-strategy bidding while the number of bidders is concealed, the maximum of $[Mq_d^*]^{\frac{\alpha-1}{\alpha}} \times \{[1 - (1 - q_d^*)^M]v - Mq_d^*\Delta\}^{\frac{1}{\alpha}}$ is achieved with concealment policy. Therefore, any contest with number of bidders being disclosed is dominated by a contest $r(\hat{q})$ with the number of bidders being concealed.

**References**


