Impulse Control of Interest Rates

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This paper examines the effect that a central bank’s interventions have on longer term interest rate securities by examining a stochastic short rate process that can be controlled by the central bank. Rather than investigate the motivations for the intervention, we assume that the bank is able to quantify its preferences and tolerances for various rates. We allow for a very general class of stochastic processes for the short rate, and most of the popular models in literature fall within this class. Interventions are best modeled as impulse controls, which are very difficult to handle, even computationally, except in very special cases. Allowing interventions to be modeled by impulse controls, we develop a computational method and provide relevant convergence results. We also derive error bounds for intermediate iterations. Using this method, we solve for the central bank’s optimal control policy and also study the effect of this on longer term interest rate securities using a change of measure. The method developed here can easily be applied to a very wide range of impulse control problems beyond the realm of interest rate models.

Subject classifications: impulse control; interest rates; free boundary problems; central bank intervention.


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1. Introduction

Interest rate securities make up a huge portion of the global financial markets. Roxburgh et al. (2011) estimates that the size of the global interest rate securities market in 2011 was $157 trillion, whereas the global equities market was only $54 trillion. The dynamics of interest rates have been heavily studied in the financial literature. Most of these studies have made the assumption that interest rates move freely in an open market. Relatively much less attention has been given to studies that consider the optimal control of interest rates by a nation’s central bank that has the power to intervene in the interest rate market. This ability to intervene in the interest rate market is seen, for example, in the United States through the Federal Reserve’s open market operations (Freund and Gutentag 1969). Here the Federal Reserve (Fed) periodically sets a target short term interest rate (Fed funds target rate) and trades various securities in large quantities to keep the short term rate close to this target. This is a common practice in many other countries as well. The Fed issued large interventions in the short term interest rate market 41 times between 2000 and 2012, 1 and in each of these instances there were large jumps in the interest rate market, indicating that it is important to investigate the Fed’s intervention policy and the way this policy affects longer term interest rates.

A central bank may want to intervene in the interest rate market for several reasons; it may wish to limit inflation to a certain level or it may want to maintain a certain exchange rate with another country. Rather than investigating the motivations for the bank’s interventions, we assume that the bank is able to precisely quantify its preferences and tolerances. Specifically, this means that the bank is able to quantify its relative tolerances for various rates above and below its preferred rate level. Also motivated by the frequency and size of observed interest rate interventions, we assume that the central bank’s aversion to intervening too often can be captured by a “cost” of intervention that the bank can quantify as fixed cost with an optional proportional component. The Fed’s objective is then to find the best intervention strategy. Our goal is to compute and examine the implications that these interventions have on the term structure of interest rates. In doing this, we find that the model is able to replicate the market’s current yield curve. We can therefore replicate the current state of the interest rate market while assuming that in the future the central bank may intervene. This possibility has not yet been seen in the interest rate literature.

This paper considers a model in which a country’s central bank can intervene in the domestic short term interest rate market. The model allows for many popular stochastic short rate models in literature. Given the costs of deviating from the target rate and the costs of control, the objective of the central bank is to find the optimal policy that strikes the best balance between frequent intervention and large deviations
from the target rate. This yields a stochastic control problem and more specifically a stochastic “impulse” control problem due to the generality of the cost structure. We model the central bank’s intervention policy as an impulse control because of the frequency and size of the Fed’s interventions. A classical control model is only appropriate when the Fed can make changes to the rate-of-change of short rate rather than bring about an instantaneous change to the short rate directly. Moreover, the size of short rate jumps are rather significant and hence a singular control model would be inappropriate as well. Of the 41 times the Fed intervened between 2000 and 2012, the average magnitude of intervention was about 14% of the Fed funds rate.

Such impulse control problems, where the controller has the ability to bring about a discontinuity in the state (the short rate) dynamics, are notoriously hard to solve. Impulse controls are natural ways to model the large economic decisions that are made infrequently but are often approximated with controls that do not bring about such discontinuity (for example, by only allowing proportional costs of control), to foster solvability. There is absolutely no hope of finding closed form solutions of impulse control problems, and all solution methods that are available are numerical. Although one can discretize the problem and then brute force the solution using value or policy iteration, these methods are very inefficient even for singular control problems. For impulse control problems, these inefficiencies are even worse. Moreover since there are several short rate models, this paper also develops an iterative method that can solve a very general class of impulse control problems and can hence easily be applied to a very wide range of impulse control problems beyond the realm of interest rate models. We provide relevant convergence results and derive error bounds for intermediate iterations.

1.1. Related Literature and Outline

Two main streams of literature are relevant: interest rate models and impulse control. In following the overview of the interest rate literature, we focus mostly on short rate models. We concentrate on short rate models because the Fed’s main avenue for intervention is in the Fed funds market, which is an extremely short term, overnight, interest rate market. Additionally, short rate models are mathematically tractable and guarantee the absence of arbitrage; see, for example, Shreve (2004, Chap. 6.5).

In a popular paper, Vasicek (1977) finds a closed form expression for bond prices when the short rate follows a simple mean reverting Ornstein-Uhlenbeck process. Cox et al. (1996) present a more robust model for the short rate that includes state dependent volatility and again finds a closed form solution for bond prices. Hull and White (1990) generalize these models and also find closed form solutions to bond prices that are able to match any existing yield curve, and Black and Karasinski (1991) consider a model for the log of the short rate. Chan et al. (1992) make an empirical comparison of several short rate models and Chapman and Pearson (2000) investigate nonlinearities in short rate models. Piazzesi (2005) examines the yield curve when the Fed funds target rate follows a compound Poisson process, Balduzzi et al. (1997) look at the effect of policy changes by the Fed on the yield curve, and Rudebusch (1995) models the behavior of the Federal Reserve’s intervention behavior and examines the effect of this on the yield curve.

We model the central bank’s ability to intervene in the interest rate market as an impulse control problem. Impulse control problems are seen, for example, in Constantinides and Richard (1978) to model a cash management problem and in Sulem (1986) to model an inventory management problem. Harrison et al. (1983) study impulse control in a canonical setting, and Feng and Muthuraman (2010) present a computational method for solving an impulse control problem in the case of a Brownian motion. Dai and Yao (2013a) prove a general property of impulse control strategies under Brownian motion. Additionally, there has been some work at the interface of these two areas. In particular, Cadenillas and Zapatero (1999) present a model in which the central bank wishes to keep exchange rates at a certain level. To achieve this goal, the bank issues an impulse control on the exchange rate; in Cadenillas and Zapatero (2000), the bank also has the ability to exactly set the interest rate. The stochastic dynamics in Constantinides and Richard (1978), Sulem (1986), Feng and Muthuraman (2010), and Dai and Yao (2013a) are restricted to a Brownian motion. The dynamics in Cadenillas and Zapatero (2000) and Cadenillas et al. (2010) are restricted to geometric Brownian motion and an Ornstein-Uhlenbeck process, respectively.

The computational method developed in this paper is a generalization of the method presented in Feng and Muthuraman (2010) that works only for the impulse control of a Brownian motion. Feng and Muthuraman (2010) leverages heavily on the restriction on Brownian motions and also on the past results that were known for the Brownian case. Popular interest rate models will not fall within the scope of Feng and Muthuraman (2010) and a generalization is needed. Although it is exciting to see that the idea of transforming a free boundary problem to a sequence of fixed boundary problems can be helpful in solving a very general class of stochastic processes, establishing the necessary convergence results in this general case is a significant challenge and more involved. Apart from the required proofs of convergence, we also present an $\epsilon$-optimality result. Since all numerical algorithms have to be stopped after convergence within a tolerance, the $\epsilon$-optimality result is extremely critical because it maps the tolerance to bounds on the objective value.

The rest of the paper is organized as follows. In §2 we present the stochastic model for the evolution of interest rates and describe the central bank’s possible intervention strategies. Section 3 describes the equation for the expected cost of control and presents an algorithm that minimizes this cost by solving a free boundary problem for a large class of stochastic processes. Then using this optimal control policy, in §4 we find the price of a zero coupon bond and
show that the model is able to capture interesting term structures with a change of measure. Section 5 shows a few examples and highlights the differences between controlled and uncontrolled short rate processes. Finally, §6 concludes.

2. The Short Rate Model

Following the convention in Vasicek (1977), we start with a market in which the federal government issues default-free zero-coupon bonds that are traded in an open market. A zero-coupon bond is a bond that pays some known quantity, say, $S_1$, when it matures with no intermediate payments, or coupons, before maturity. At time $t$, we say the price of such a bond that matures at time $T > t$ is $B(t,T)$, also called the discount factor. The yield to maturity of this bond is the value, $R(t,T)$, that satisfies

$$B(t,T) = e^{-R(t,T)(T-t)}.$$  \hfill (1)

The short rate, $r_t$, the instantaneous rate of interest is given by

$$r_t = \lim_{t \downarrow t} R(t,T).$$  \hfill (2)

As in Vasicek (1977) and Cox et al. (1996), we assume that the short rate follows a stochastic process; however, we incorporate a very general model for the stochastic process, and we allow the central bank to control it. We model the uncontrolled short rate as a general stochastic process, under the physical measure, described by

$$d\hat{r}_t = \mu(\hat{r}_t)dt + \sigma(\hat{r}_t)dW_t. \hfill (3)$$

This general form allows for several common stochastic processes. For example, if $\mu(r) = a - b \cdot r$ and $\sigma(r) = \sigma_0 \sqrt{r}$, then Equation (3) is the familiar mean reverting square root model used to model the short rate in Cox et al. (1996). Additionally, if $\sigma(r) = \sigma_0$ then Equation (3) is an Ornstein-Uhlenbeck process used to model the short rate in Vasicek (1977). Arithmetic and geometric Brownian motion are also both possible in this framework.

The central bank is then able to apply an impulse control to the short rate, which allows it to instantaneously move the short rate up or down by some nonzero amount. An impulse control, $\nu$, is defined as a sequence of nondecreasing stopping times $\{\tau_j\}_{j=1}^\infty$ associated with the corresponding amounts of control $\{\xi_j\}_{j=1}^\infty$. Given an impulse control $\nu = (\tau_1, \xi_1; \ldots; \tau_i, \xi_i; \ldots)$, the controlled short rate becomes

$$\begin{cases}
    dr_t = \mu(r_t)dt + \sigma(r_t)dW_t, & \tau_j \leq t < \tau_{j+1}, \\
    r_{\tau_j} = r_{\tau_j} + \xi_j.
\end{cases} \hfill (4)$$

As mentioned earlier, we assume that the central bank has the ability to quantify its preferences and tolerances for the level of the short rate. Specifically, this implies that the central bank is able to quantify its desire for the short rate to remain at a target rate with a running cost function and its aversion to intervention through fixed and proportional costs. These costs do not necessarily represent money paid by the central bank; rather, the running cost reflects the bank’s desire for the short rate to be at a certain target level, and the control costs represent the bank’s aversion to intervening in the interest rate market, similar to the policymaker’s objective in Lohmann (1992) and Cadenillas and Zapatero (1999). The central bank must weigh these two goals and make a decision about how best to control the interest rate market. The central bank’s goal is to find the optimal control policy, $\nu$, to minimize these costs, but in order to do this, we must first define the cost structure that encompasses running costs and control costs.

At each moment, $t$, we say the economy with short rate $r_t$ incurs a running cost at a rate of $h(r_t) \geq 0$, where $h$ is designed to penalize the bank for deviations from the target short rate; for example, if the central bank’s target short rate is 1%, then $h$ could be increasing above and decreasing below 1% to represent the desire for the rate to stay at 1%. The central bank also incurs a cost, $G(\xi)$, each time it applies a control of size $\xi$ to the short rate. This represents the central bank’s desire to avoid intervening too often in the market. The cost of control has two components, a fixed component and an optional proportional component. This means that each time the government increases or decreases the short rate it suffers a fixed cost as well as a cost proportional to the amount by which it moves the short rate. It is apparent that central banks do not want to intervene in the interest rate market often, and for this reason it makes sense that there is a perceived fixed cost for each time the bank exerts control. We define the cost of control as

$$G(\xi) = \begin{cases}
    K + k \cdot \xi & \text{if } \xi \geq 0, \\
    L - l \cdot \xi & \text{if } \xi < 0.
\end{cases} \hfill (5)$$

where $K, L, k, l > 0$. Here we can see that the cost of control is an asymmetric function of $\xi$; the fixed cost of increasing or decreasing the short rate are $K$ and $L$, respectively, and the proportional cost of increasing or decreasing the short rate by $\xi$ are $k$ and $l$, respectively. We consider an infinite planning horizon and we discount future costs at a rate $\beta > 0$. Putting all this together, the central bank’s objective is to pick a control policy to minimize the expected value of future discounted costs. The total expected cost of using control $\nu$ is given by

$$J(\nu) = \mathbb{E}\left[ \int_0^\infty e^{-\beta t} h(r_t)dt + \sum_n e^{-\beta \tau_n} G(\xi_n) \mid \tau_0 = r_0 \right]. \hfill (6)$$

In order for this to be a well-defined problem, we only consider those impulse controls, $\nu$, such that $\forall r < \infty$, $J(\nu) < \infty$ as in Harrison et al. (1983). We also place some technical restrictions on $h$ to avoid trivial solutions. The restrictions on $h$ are detailed in the following assumption.
ASSUMPTION 1. The function \( h(\cdot) \) is a nonnegative function, it is continuously differentiable with the only possible exceptions at a finite set \( N_\mu \), and there exists a point \( x_0 \) such that
- \( h(r) \) is nonincreasing for all \( r < x_0 \);
- \( h(r) \) is nondecreasing for all \( r > x_0 \).

Also, there exist some points \( z_1 \) and \( z_2 \) such that
- \( h'(r) > (\beta - \mu'(r)) \cdot l \) for all \( r > z_1 \);
- \( h'(r) < (\beta - \mu'(r)) \cdot k \) for all \( r < z_1 \).

Furthermore,
- \( h'(r) \geq (\beta - \mu'(r)) \cdot l \) implies \( h(z) \geq (\beta - \mu'(r)) \cdot l \) for all \( z > r \);
- \( h'(r) \leq (\beta - \mu'(r)) \cdot k \) implies \( h(z) \leq (\beta - \mu'(r)) \cdot k \) for all \( z < r \).

The intuition of Assumption 1 is that when the short rate becomes too large or too small, the running cost \( h \) will grow fast enough that it is better to intervene the short rate. Assumptions on \( h \) made in Constantinides and Richard (1978), Feng and Muthuraman (2010), and Dai and Yao (2013b) are special cases of Assumption 1.

DEFINITION 1. An impulse control \( \nu = (\tau_1, \xi_1; \ldots; \tau_i, \xi_i; \ldots) \) is called admissible if the following conditions are satisfied:
- \( |\xi_i| > 0, \tau_i < \tau_{i+1} \), and \( \xi_i \) is \( \mathcal{F}_\tau \) measurable for every \( i \);
- \( f_i(\nu) < \infty \) \( \forall \nu < \infty \);
- The controlled state process \( r_t \) satisfies the growth conditions.

\[
\lim_{t \to \infty} e^{-\beta t} \mathbb{E}[|r_t|] = 0,
\]

\[
\mathbb{E} \left[ \int_0^\infty e^{-2\beta t} \sigma^2(r_t) \, dt \right] < \infty.
\]

The collection of all the admissible impulse controls is denoted \( \mathcal{A} \).

Equations (7) and (8) guarantee that the controlled short rate does not grow uncontrollably and are essentially mild technical conditions similar to the ones in Korn (1997).

3. The Value Function

The bank’s objective is to find a control, \( \nu \), such that the associated cost function \( f_i(\nu) \) is minimized. We call the cost optimal function corresponding to such a control the value function,

\[
V(r) = \min_{\nu \in \mathcal{A}} f_i(\nu).
\]

Bensoussan and Lions (1973) use quasi-variational inequalities (QVI) to study stochastic impulse control problems. Thereafter, QVIs have become a standard way of formulating impulse control problems. The main goal of QVI is to adopt dynamic programming arguments with Itô’s formula to characterize the value function and the optimal control using a differential equation problem. The QVI is derived using the idea that if at time zero we use some control for an infinitesimal amount of time, and then immediately switch to the optimal control, the resulting cost cannot be less than the cost function associated with the optimal control applied since the very beginning. We now develop the QVI for the problem considered here with a brief explanation.

First, let \( \mathcal{A} \) be the infinitesimal generator of the uncontrollable process so that

\[
\mathcal{A}v(r) = \frac{1}{2} \sigma^2(r) \nu'(r) + \mu(r) \nu'(r).
\]

Then, assuming enough smoothness, we have \( \mathcal{A}V(r) - \beta V(r) + h(r) \geq 0 \) (a.e.). This is the result of the aforementioned dynamic programming argument: no intervention is placed during \( t \in [0, \Delta t] \) for some small \( \Delta t > 0 \), and then we switch to the optimal policy thereafter, so the resulting cost function will be no better than the optimal; letting \( \Delta t \) go to zero and applying Itô’s formula gives us this result.

Also because of the dynamic programming argument, for any given \( r \) and \( \xi \), we have \( V(r + \xi) + G(\xi) - V(r) \geq 0 \). This means if at time zero we place some intervention (to increase or reduce the short rate process) and then switch to the optimal control policy, the resulting cost function will be at best equal to the optimal one. Taking infimum over all possible \( \xi \) yields \( \inf_{\xi} V(r + \xi) + G(\xi) - V(r) \equiv \inf_{\xi} V(r) \geq 0 \).

The dynamic programming argument implies that one of these inequalities should be tight for each value of \( r \), depending upon what is optimal. Putting all this together, we have the following QVIs

\[
\begin{align*}
\mathcal{A}V(r) - \beta V(r) + h(r) & \geq 0, \quad \text{a.e. } r, \\
\inf_{\xi} V(r + \xi) + G(\xi) - V(r) & \equiv \inf_{\xi} V(r) \geq 0, \quad \forall r, \\
\mathcal{A}V(r) \cdot [\mathcal{A}V(r) - \beta V(r) + h(r)] = 0, \quad \forall r.
\end{align*}
\]

Hereafter, we refer to (10) as the QVI for the impulse control problem considered. Theorem 1 says that if a cost function \( v \) associated with an admissible impulse control policy \( \hat{\nu} \) satisfies the QVI (10) together with some technical conditions, then \( v \) coincides with the value function \( V \).

THEOREM 1 (VERIFICATION THEOREM). Suppose \( v(r) \), the cost function associated with an admissible impulse control policy \( \hat{\nu} \), satisfies the QVI (10). If the following are satisfied
1. \( v \) is linear for \( r \leq d \) and \( r \geq u \) for some \( d < u \),
2. \( v \in C^1(\mathbb{R}) \cap C^2(\mathbb{R}\setminus [d, u]) \),
then \( v \) coincides with the value function \( V \).

The proof can be found in the online appendix (available as supplemental material at http://dx.doi.org/10.1287/opre.2014.1270).

Because of the dynamic programming argument, in the region \( C = \{ r \colon \mathcal{A}V(r) - \beta V(r) + h(r) = 0 \} \), the optimal decision is not to intervene. Such a region, \( C \), is called the continuation region. As soon as the process leaves the continuation region, we have \( \mathcal{A}V(r) \neq 0 \), and thus the control implied by the QVI chooses to intervene by increasing or decreasing the short rate by the appropriate amount. We next describe the differential equation problem that will enable us to find the continuation region as well as the value function.
3.1. The Free Boundary Problem

The value function, $V$, which solves the QVI, partitions $\mathbb{R}$ into two disjoint sets, the continuation region and the intervention region. To search for the optimal policy is essentially to search for boundaries of the continuation region and the size of the intervention when the short rate hits the boundaries of the continuation region. Such a problem, wherein the boundaries must be computed as part of the solution, is called a free boundary problem.

A control band policy is a policy that can be characterized by four points, $d, D, U, u$ with $d < D \leq U < u$. The corresponding strategy is to exert intervention if the short rate attempts to exit $(d, u)$ and the process is left uncontrolled when in $(d, u)$. When the short rate strikes or is below $d$, the control policy instantaneously increases it to $D$; similarly, when the rate strikes or is above $u$, the policy instantaneously decreases it to $U$. This policy is illustrated in Figure 1. In this paper we will focus exclusively on control band policies and we will show, by construction, that the optimal policy is a control band policy.

Now, given any admissible impulse control policy, $v$, characterized by a control band $(d, D, U, u)$, Theorem 2 shows that we obtain its cost function $J_r(v)$ by solving the following second-order differential equation problem with fixed boundary

$$0 = \beta v(r) - \beta \cdot v(r) + h(r), \quad \forall r \in (d, u), \quad (11)$$

$$v(d) = v(D) + k \cdot (D - d), \quad (12)$$

$$v(u) = v(U) + l \cdot (u - U). \quad (13)$$

**Theorem 2.** Suppose $v$ is an admissible impulse control characterized by $(d, D, U, u)$. If $v(r) \in C^2(d, u)$ solves the differential equation problem $(11)-(13)$ in $[d, u]$, then $v(r)$ is equal to the cost function $J_r(v)$ associated with $v$ for $r \in [d, u]$.

The proof can be found in the online appendix.

By the definition of the $(d, D, U, u)$ policy, we can see that

$$J_r(v) = \begin{cases} 
  v(D) + K + k \cdot (D - r) & \text{if } r \leq d, \\
  v(d) + k \cdot (d - r) & \\
  v(U) + l \cdot (r - U) & \text{if } r \geq u, \\
  v(u) + l \cdot (r - u) &
\end{cases} \quad (14)$$

This enables us to obtain the cost function associated with $(d, D, U, u)$ for every possible $r$ by extending $v$ from $(d, u)$ to the whole real line. Hereafter, we denote $v$ as the solution to Equations (11)–(13) together with the extension in Equation (14).

Now, given a control policy, $v$ described by $(d, D, U, u)$, we solve the boundary value problem described in Equations (11)–(13) to compute $v$ and extend it with (14). We then use Theorem 2 to verify that this is equal to $J_r(v)$. This means that finding the optimal control policy is the same as finding the $(d, D, U, u)$ such that the resulting $v$ satisfies the optimality conditions in Equation (10), meaning $V$ is the solution to a free boundary problem.

In the next section we describe an efficient method to find the optimal policy corresponding to the solution to the free boundary problem.

3.2. Finding the Free Boundary

In order to solve the free boundary problem from the previous section, we now describe an iterative algorithm that converts the free boundary problem into a sequence of fixed boundary problems, and we restrict our attention to control band policies. For some given control band policy characterized by $(d_0, D_0, U_0, u_0)$ we find the associated cost function as the solution to Equations (11)–(14), and call this $V_0$. Then with this policy and cost function the algorithm finds a new policy and cost function, $(d_{n+1}, D_{n+1}, U_{n+1}, u_{n+1})$ and $V_{n+1}$, that is closer to the optimal control policy and value function. Upon iteration we find that this algorithm monotonically converges to the optimal control policy and value function.

To begin the algorithm we start with an initial guess policy $(d_0, D_0, U_0, u_0)$ with $d_0 < D_0 \leq U_0 < u_0$. We obtain the associated cost function by solving the fixed boundary problem (11)–(14) and denote it $V_0$. The choice of this initial guess is not entirely trivial because we must be sure that the optimal continuation region is contained within $(d_0, u_0)$. We can check this property with the following condition,

$$\lim_{r \downarrow d_0} V_0(r) + k \geq 0, \quad (15)$$

$$\lim_{r \uparrow u_0} V_0(r) - l \leq 0. \quad (16)$$

We say that an initial guess satisfies the superset condition if the above inequalities hold. An initial guess can be found easily by, for example, increasing the length of $(d_0, u_0)$, perhaps by 50%, repeatedly until the superset condition is satisfied. Intuitively, if the superset condition is not
satisfied, say, \( \lim_{r \to d_0} V'_n(r) + k < 0 \), then it indicates that in the neighborhood above \( d_0 \), the cost function decreases faster than the cost of control. However, when \( r \) is sufficiently small, by assumption the running cost grows fast enough to justify the exertion of control to increase the short rate. This means, when \( d_0 \) is small enough, the proportional control cost decreases faster than the cost function in the neighborhood of \( d_0 \), which will make the superset condition at \( d_0 \) satisfied. An analogous argument holds for sufficiently large \( u_0 \). A similar explanation of the assumptions made on the running cost function, \( h \), can be found in Constantinides and Richard (1978), Feng and Muthuraman (2010), and Dai and Yao (2013b). Once such an initial guess of \((d_0, D_0, U_0, u_0)\) are found, we begin our iteration by setting \( n = 0 \).

The next step of the algorithm is to find new values of \( d \) and \( u \). We define these new values as

\[
d_{n+1} = \sup \{y \in [d_n, D_n]: \forall r \in [d_n, y], V'_n(r) + k \geq 0 \},
\]

\[
u_{n+1} = \inf \{y \in (U_n, u_n]: \forall r \in [y, u_n], V'_n(r) - l \leq 0 \}.
\]

We illustrate the update procedure for \( d \) in Figure 2. The solid black line in the figure represents the cost function on an iteration. We see that between \( d_n \) and \( d_{n+1} \), we have \( V'_n(r) + k \geq 0 \) and that the slope at \( d_{n+1} \) is equal to \(-k\). We also illustrate the boundary condition from Equation (12) here that relates the values of the cost function at \( d_n \) and \( D_n \).

We select \( d_{n+1} \) in this way because in a region above \( d_n \) if we have \( V'_n \geq -k \), then the proportional cost decreases at least as fast as the value function, suggesting it would be beneficial to increase \( d \). Also, we know that at convergence we want the first derivative to be continuous across \( d \), as required by Theorem 1, so we pick \( d_{n+1} \) to help satisfy that constraint. Similar reasoning also holds for \( u_{n+1} \). We can see in this figure that the slope of \( V_n \) at \( d_{n+1} \) is equal to \(-k\). This helps us see that at convergence, the value function should have a continuous first derivative, otherwise known as the smooth pasting principle.

After we find the new values \( d_{n+1} \) and \( u_{n+1} \) we find the cost function associated with the control band policy characterized by \((d_{n+1}, D_{n+1}, U_{n+1}, u_{n+1})\) by re-solving Equations (11)–(13). We call this cost function \( V_{n+1}(r) \). With this new cost function, \( V_n(r) \), we define new values of \( D \) and \( U \) as

\[
D_{n+1} = \arg \min_{r \in (d_{n+1}, u_{n+1})} \{V_n(r) + k \cdot r\},
\]

\[
U_{n+1} = \arg \min_{r \in (d_{n+1}, u_{n+1})} \{V_n(r) - l \cdot r\}.
\]

Later we will show that \( D_{n+1} \leq U_{n+1} \). \( D_{n+1} \) and \( U_{n+1} \) are chosen to be the most efficient “jump to” points for the given associated cost function. With \((d_{n+1}, D_{n+1}, U_{n+1}, u_{n+1})\) we can finally find \( V_{n+1}(r) \) by solving Equations (11)–(13) with these new boundary locations and then iterate this until convergence. We present a flowchart that illustrates this algorithm in Figure 3.

Theorem 3 shows that for any \( r \) we have \( V_{n+1}(r) \leq V_n(r) \) and the superset condition holds with the new \( V_{n+1}, d_{n+1}, u_{n+1} \). These conditions warrant the repetitive improvement on the cost function as well as a sequence of shrinking continuations regions \((d_n, u_n)\).

**Theorem 3.** Given an admissible impulse control policy characterized by \((d_n, D_n, U_n, u_n)\), let \( V_n(r) \) be defined by the solution to Equations (11)–(14). If \( V_n \) and \( (d_n, D_n, U_n, u_n) \) satisfy the superset condition, let \( d_{n+1}, u_{n+1}, V_{n+1}, D_{n+1}, U_{n+1} \) and \( V_{n+1} \) be as in the above algorithm; then we have

1. \( V_n(r) \leq V_n(r), \forall r \);
2. \( \lim_{r \to d_{n+1}} V'_n(r) + k \geq 0, \lim_{r \to u_{n+1}} V'_n(r) - l \leq 0; \)
3. \( V_{n+1} \leq U_{n+1}; \)
4. \( V_{n+1} \leq V_n(r), \forall r; \)
5. \( V_{n+1} \) satisfies the superset condition with \( d_{n+1}, u_{n+1}, \), which means \( \lim_{r \to d_{n+1}} V_{n+1}(r) + k \geq 0 \) and \( \lim_{r \to u_{n+1}} V_{n+1}(r) - l \leq 0. \)

The proof of this theorem can be found in the online appendix.

Theorem 3 establishes that the boundary update procedure can be iteratively used to improve the cost function monotonically and to shrink the continuation region; thus the scheme is guaranteed to converge. Figure 4 gives an illustration of a sequence of \( V_n \). The sequence is monotonically decreasing and the dots on each curve represent \((d_n, D_n, U_n, u_n)\). Each \( V_n \) in the sequence is not expected to be in \( C^1 \) over the whole space; however, the \( v \) associated with the converged \((d, D, U, u)\) policy is expected to have a continuous derivative everywhere because of the smooth pasting nature of the boundary update Equations (17) and (18). The following

---

**Figure 2.** An illustration of updating \( d \) and Equation (12).
Figure 3. Description of boundary update algorithm.

Theorem 4. Suppose that \( h(r) \) satisfies Assumption 1 and let \((d, D, U, u)\) be the policy obtained at convergence with \( v \), its associated cost function. This implies that \( v(r) \) solves (11)–(14) and that \( v \) is \( C^1 \). If \( \beta - \mu'(r) \geq 0 \) \( \forall r \), then it is identical to the optimal value function \( V \). If, however, \( \beta - \mu'(r) < 0 \) for some \( r \), and if \( h'(d) + (\beta - \mu'(d)) \cdot k \leq 0, -k < \psi'(r) < 1 \) in \((D, U)\), \( \psi(r) + k \leq 0 \) in \([d, D]\), \( -\psi(r) + l \leq 0 \) in \([U, u]\), and \( h'(u) + (\mu'(u) - \beta) \cdot l \geq 0 \), then \( v(r) \) is identical to the value function \( V(r) \), and \((d, D, U, u)\) is the corresponding optimal control policy.

The proof is given in the online appendix.

We can see that popular short rate models such as the Vasicek and the CIR model have \( \mu'(r) < 0 \), which falls into the case \( \beta - \mu'(r) \geq 0 \). Therefore, the optimality of the solution obtained by the monotone improvement scheme is warranted. For general diffusion process with drift that does not necessarily satisfy \( \beta - \mu'(r) \geq 0 \) \( \forall r \), we have provided additional conditions for optimality in Theorem 4.

Since any computational iteration must be stopped when a specific tolerance is reached, the question of how the tolerance relates to deviations from optimality is important. The following theorem and corollary provide an upper bound on the difference between the optimal value function and the cost function associated with the \((d, D, U, u)\) policy obtained in any step of the scheme. It is also important to note that whereas the convergence to optimality established in Theorem 4 requires some conditions on \( \beta - \mu'(r) \), the \( \epsilon \)-optimality result that follows does not make any assumptions on \( \mu'(r) \).

Theorem 5 (\( \epsilon \)-Optimality). Suppose \( v(r) \) is the cost function associated with some admissible impulse control characterized by \( d < D \leq U < u \). If \( v(r) \) satisfies the following conditions for some \( \epsilon_1, \epsilon_2, \epsilon_3 > 0 \)

\[
\begin{align*}
\min_{\xi > 0} &\{v(r + \xi) + K + k \cdot \xi - v(r)\} \\
\min_{\xi > 0} &\{v(r - \xi) + L + l \cdot \xi - v(r)\}
\end{align*}
\]

\[
\left\{\begin{array}{l}
\Delta v(r) - \beta \cdot v(r) + h(r) \geq -\epsilon_1, \quad \text{a.e. } r, \\
\inf_{\xi > 0} [v(r + \xi) + K + k \cdot \xi] - v(r) \geq -\epsilon_2, \\
\inf_{\xi > 0} [v(r - \xi) + L + l \cdot \xi] - v(r) \geq -\epsilon_3,
\end{array}\right.
\]

then we have

\[
v(r) \leq \left(1 + \frac{\epsilon_1}{K} + \frac{\epsilon_2}{L}\right) \cdot V(r) + \frac{\epsilon_3}{\beta},
\]

in which \( V(r) \) is the value function.

The detailed proof is found in the online appendix.

Corollary 1. Suppose \( v(r) \) is the cost function associated with some admissible impulse control characterized by
where \( d < D \leq U < u \). If \( v(r) \) satisfies the following conditions for some \( \epsilon > 0 \):
\[
\begin{align*}
\frac{d}{dt}v(r) - \beta \cdot v(r) + h(r) &\geq -\epsilon, \quad \text{a.e. } r, \\
\mathbb{E}[v(r)] &\geq -\epsilon,
\end{align*}
\]
then we have
\[
v'(d+) + k \geq 0, \quad -v'(u-) + l \geq 0,
\]
in which \( \tilde{K} \equiv \min(K, L) \).

The detailed proof can be found in the online appendix.

4. Bond Prices

This section focuses on pricing a bond whose short rate follows the optimally controlled dynamics. In the previous sections all computations were performed by the central bank under the physical measure because the bank is concerned with the actual deviations from the target short rate. Now, however, we must abandon this measure and use the risk-neutral measure to find bond prices because this prevents arbitrage in the market.

We first note that the price of a bond is determined by the bond’s tenor and the current short rate; in this way we can rewrite the price of a bond as \( B(t, T, r) \). To find the price of the bond, we must use the Girsanov theorem to change the physical measure to the risk-neutral measure. In doing this we can rewrite the controlled dynamics between the stopping times that correspond to \( d \) and \( u \) as
\[
dr_r = \tilde{\mu}(t, r) dt + \sigma(r) dW_t^\mathbb{Q}, \quad \tau_i \leq t < \tau_{i+1},
\]
where \( \tilde{\mu}(t, r) \) is determined by the market and \( W_t^\mathbb{Q} \) is a Brownian motion under the risk-neutral measure. Given this representation the price of a bond is given by
\[
B(t, T, r) = \mathbb{E}^\mathbb{Q} \left[ e^{-\int_t^T \tilde{\mu} \, dt} \left| r_T = r \right. \right],
\]
where the expectation is taken with respect to the risk-neutral measure, and the short rate process that is being evaluated includes the impulse controls administered by the central bank. This is the typical bond pricing formula; however, it is not very useful in our setting because we do not know \( \tilde{\mu}(t, r) \). We therefore state the following theorem to find the appropriate change of measure and price a bond based on the controlled short rate.

**Theorem 6 (Price of a Bond).** The price of a bond whose short rate follows the optimally controlled dynamics described in the previous section satisfies the following PDE with boundary conditions
\[
\begin{align*}
\frac{\partial B}{\partial t} + (\mu(r) + \sigma(r) \cdot q(t, r)) \frac{\partial B}{\partial r}
\end{align*}
\]
\[
+ \frac{1}{2} \sigma^2(r) \frac{\partial^2 B}{\partial r^2} - r \cdot B = 0, \quad t < T, \quad r \in (d, u),
\]
(23)
\[
B(T, T, r) = 1, \quad \forall r,
\]
(24)
\[
B(t, T, d) = B(t, T, D), \quad \forall t \leq T,
\]
(25)
\[
B(t, T, u) = B(t, T, U), \quad \forall t \leq T,
\]
(26)
where \( d, D, U \) and \( u \) describe the optimal control policy, and \( q(t, r) \) is the so-called market control policy, and \( h(r) \) is the so-called market control policy, and \( q(t, r) \) is the so-called market control policy.

The proof of this theorem, which relies on an arbitrage argument and closely follows Vasicek (1977), is found in the appendix.

The market price of risk, \( q(t, r) \), found in Equation (23), represents the instantaneous trade-off between the expected return of a bond per unit of volatility and is to be observed by the market. Given this market price of risk, we can immediately recognize that \( \tilde{\mu}(t, r) = \mu(r) + \sigma(r) \cdot q(t, r) \). If we take \( \sigma(r) \) to be constant and \( q(t, r) \) to be a function of only time, then this can be a variant of the model found in Hull and White (1990). This model has a closed form solution for Equations (23) and (24) without the boundary conditions (25) and (26), which is the solution to Equation (22) without control. However with these boundary conditions, there is no closed form solution, and thus any solution to the PDE with boundary conditions must be found numerically.

A convenient property of this model is that given the controlled dynamics of the short rate, we can extract the market price of risk from the current yield curve, where the relationship between bond prices and the yield curve is defined in Equation (1). In fact, we are able to generate almost any desired yield curve by finding the appropriate market price of risk.

Under this model it is also possible to consider a market price of risk that is a function of both time and the short rate. We can select \( \mu(r) \) and \( q(t, r) \) in such a way that under the risk-neutral measure the dynamics of the short rate can be represented as
\[
dr_r = (\tilde{a}(t) - \tilde{b}(t)r) dt + \sigma(r) dW_t^\mathbb{Q}.
\]
This means that we can represent the short rate as a mean reverting process with time varying mean and mean reversion rate, if \( \tilde{b}(t) > 0 \). In this model we can still perfectly replicate the market’s yield curve and we can also replicate the interest rate derivative market as in Hull and White (1990).

5. Analysis of Results

In the previous sections we described how to find the optimal intervention policy and the yield curve based on that intervention policy. We now present a few examples that use these results to highlight the differences between a controlled and an uncontrolled short rate process. In our first example we take the uncontrolled short rate process to be a mean reverting square root process, as in Cox et al. (1996), described by \( d\tilde{r}_t = \lambda(\theta - \tilde{r}_t) dt + \sigma \sqrt{\tilde{r}_t} dW_t \), where we set \( \lambda = 1, \theta = 0.07, \sigma = 0.12 \). Additionally we set the costs of control to be \( K = 0.005, k = 0.18, L = 0.005, l = 0.12, \beta = 0.01 \). For the sake of illustration we take the running cost function to be \( h(r) = 0.07 \log(r/\theta)^2 + (\theta - r)^2 \). This choice of nonlinear running cost function is motivated by Cadenillas and Zapatero (2000). It captures the preference to keep rates
well above zero and also the central bank’s preference to keep the rate close to a target \( \theta \). This running cost function also satisfies the technical growth conditions, from Assumption 1, to ensure that the formulation is meaningful.

Under this short rate process and cost structure, we can find the optimal control policy using the method discussed in §3.2. In Figure 4 we plot the sequence of \( V_n(r) \) given an initial guess of \((d_0, D_0, U_0, u_0) = (0.005, 0.08, 0.12, 0.25)\). Here the cost functions, \( V_n(r) \), are getting smaller in each iteration and the dots represent the intervention policy associated with each iteration of the algorithm. The algorithm converges to the optimal value function at the bottom of Figure 4 and the optimal policy is described by \((d, D, U, u) = (0.0334, 0.0614, 0.0853, 0.1343)\). Although one can discretize the problem using the finite difference schemes proposed in Kushner (1976) and then brute force the solution using value or policy iteration, this methodology is not expected to perform well on impulse control problems because of the discontinuities in the state evolution that translate to insufficient smoothness in the value function for accelerated convergence. However, for illustration, comparing our method with that of using policy iteration on the discretized Markov chain, our method converges after 22 iterations in 0.176 seconds on a 10,000 point grid, whereas policy iteration in the controlled Markov chain case converges after 703 iterations in 1,328 seconds on the same grid. Not only is the number of iterations much larger for the controlled Markov chain, the time spent for each iteration is larger, too, resulting in almost four orders of magnitude difference.

In order to compute the yield curve, we next use the optimal intervention strategy to solve the bond pricing problem found in Theorem 6. For this example we take the market price of risk, \( q \), to be zero. With the solution to this problem, we can then obtain the yield curve of interest rates based on the controlled short rate model. We plot the resulting yield curves for two initial values of the short rate in Figure 5. In this figure we plot the yield curve resulting from both the controlled and uncontrolled short rate process; the solid lines represent the yield curve of the controlled short rate process, and the dashed lines represent the yield curve of the uncontrolled short rate. There is a closed form solution to bond prices resulting from this uncontrolled short rate process found in Cox et al. (1996).

Plot (a) in Figure 5 shows the two yield curves when the initial value of the short rate is 4.5%. In this scenario the yield curve of the controlled short rate process is above the yield curve of the uncontrolled short rate. This is because when \( r \) is close to \( d \) the probability of the central bank intervening soon is increased. This means we expect the short rate to be elevated to \( D \) soon and thus the yield on longer term bonds that are controlled by the bank must have a higher yield than do bonds that are not controlled by the bank. Similarly in plot (b) the yield curve of the controlled short rate is below that of the uncontrolled short rate because when \( r \) is close to \( u \) to probability of the central bank pushing the short rate down to \( U \) soon is high.
The yield curves in Figure 5 consider high and low initial interest rates and in Figure 6 we examine the yield curve when the initial rate is equal to \( \theta \), the long-term mean of the CIR process. In this figure plot (a) shows the yield curve for the problem described above. In this scenario we see that while the yield curve of the uncontrolled process is decreasing, the yield curve of the controlled process is increasing. This is a more drastic difference than the scenarios seen in Figure 5 because the shape of the yield curves here is different for the controlled and uncontrolled processes. In general the yield curve for an uncontrolled mean reverting process should be decreasing when the initial value is equal to the long-term mean because of the convexity in Equation (22). We find, however, that by issuing control the central bank can force this yield curve to be increasing. This could be due to the asymmetry in the optimal control band policy. In particular, since the running cost function is steep for small values of \( r \), the optimal policy issues control quickly when the process goes below the long-term mean. In contrast, the running cost function is less steep for larger values of \( r \), and the optimal policy allows the process to go relatively high before issuing control. With this in mind, we can see that the controlled short rate process will spend more time, on average, above \( \theta \) than below resulting in an increasing yield curve in this case.

To confirm that this increasing yield curve is due to the asymmetry in the control band policy, we next consider a control policy where the optimal \( d \) and \( u \) are symmetric around \( \theta \) and so are \( D \) and \( U \). The resulting yield curve is seen in plot (b) of Figure 6. In this plot we see that the yield curve of the controlled process is also decreasing. It is, however, decreasing faster than the yield curve of the uncontrolled process, again due to the convexity in Equation (22) and the asymmetry of the CIR volatility. This example displays an important case where the central bank’s intervention can change not only the level of the yield curve but also the shape.

With everything else kept the same from the first example, we also consider another example with two different values of \( \sigma \). Figure 7 displays the controlled and uncontrolled yield curves when the initial short rate is 4% for the scenarios when \( \sigma = 0.08 \) and \( \sigma = 0.14 \).

In Figure 7 the yield curves from the uncontrolled short rate processes are close together, indicating that the volatility does not affect the yield curve very much here. On the other hand, the yield curves from the controlled short rate processes differ much more for different values of \( \sigma \). This is because for the controlled short rate process \( \sigma \) affects the optimal intervention policy and thus the boundary conditions in Equations (25)–(26). For different control band policies the probabilities of intervention soon can change drastically, and this is seen in the resulting yield curves. In this example the probability of an intervention soon is higher for \( \sigma = 0.14 \).
than for $\sigma = 0.08$, and thus the yield curve is also higher in this case.

In order to understand the model's dependence on volatility better, we now examine the optimal control policy for different values of $\sigma$. In doing this we look at three popular short rate models, the Vasicek, the CIR, and the model found in Black and Karasinski (1991). The SDE for the uncontrolled short rate process in the Vasicek model is given by $d\hat{r}_t = \lambda(\hat{\delta} - \hat{r}_t)dt + \phi dW_t$ and the SDE for the Black-Karasinski model is given by $d\hat{r}_t = \lambda(\mu - \log(\hat{r}_t))\hat{r}_t dt + \gamma\hat{r}_t dW_t$. In these two models we select the parameters so that the first two moments of the long-term distribution of the uncontrolled short rate processes are the same as for the uncontrolled CIR model. In each of the three models, we use the same mean reversion parameter $\lambda$. For the Vasicek model this means we set $\delta = \theta$ and $\phi = \sigma^2/2\lambda$, and for the Black-Karasinski model we set

$$\mu = \log(\theta) + \frac{1}{2} \log\left(\frac{\sigma^2}{2\lambda \theta} + 1\right), \quad \gamma = \sqrt{2\lambda \log\left(\frac{\sigma^2}{2\lambda \theta} + 1\right)},$$

where $\theta$ and $\sigma$ are from the CIR model. In all three of these models with parameters chosen this way, the long-term mean and variance of the uncontrolled short rate are given by $E[\hat{r}_T] = \theta$ and $\text{Var}[\hat{r}_T] = \frac{1}{2} \sigma^2 \theta / \lambda$. We note that although these three models have equal average volatility, in the Vasicek model the instantaneous volatility is equal for all values of $\hat{r}_t$; however, for the CIR and Black-Karasinski models the instantaneous volatility is asymmetric in $\hat{r}_t$. In order for these volatilities to average out to the same, the CIR and Black-Karasinski models have smaller instantaneous volatilities for small values of the short rate and larger instantaneous volatilities for large values of the short rate. In addition to this asymmetry in volatility, the Black-Karasinski model also has asymmetry in its drift term, making the Black-Karasinski model the most asymmetric of the three.

In Figure 8 we show the optimal intervention policy for these three models, as a function of $\sigma$, when $\lambda$ and $\theta$ are the same as in the previous examples. Here we have also changed the cost structure and set $K = L = 0.05$, $k = l = 0.15$, $\lambda = 1$, $\theta = 0.07$, $\beta = 0.01$, and $h(r) = 20(r - \theta)^2$. We have chosen cost functions like this so that the cost of control is completely symmetric about $\theta$; that is, it costs the same to issue positive and negative control, and the running cost is a symmetric function about $\theta$. We can see in Figure 8 that the control policy for the Vasicek model is symmetric about $\theta$.

**Figure 8.** Control band policy as a function of $\sigma$ for three different models.

### Notes
Plots (a) and (b) show how far from $\theta$ the short rate must be before control is issued. Plots (c) and (d) show how much control is issued each time a boundary is hit.
In plots (a) and (b) we see that the solid black line is the same in both plots, indicating that for the Vasicek model the central bank allows the short rate to deviate equally above and below \( \theta \) before intervening. Also in plots (c) and (d), the solid black lines are the same, indicating that when the short rate process hits \( d \) or \( u \) in the Vasicek model, the central bank intervenes equally. This symmetry, however, is not present for the CIR or the Black-Karasinski model. This is because in the Vasicek model the uncontrolled short rate is Gaussian and therefore symmetric, whereas in the CIR model the short rate follows a noncentral \( \chi^2 \) distribution and the Black-Karasinski model follows a log-normal distribution. Both of these distributions are asymmetric, and therefore the central bank must account for this asymmetry when deciding its optimal control policy for the short rate.

For all three of these models, we see that for larger values of \( \sigma \) the short rate must be further away from \( \theta \) before the central bank intervenes. This can be seen in plots (a) and (b) in Figure 8; the curves are increasing with volatility in all cases. The reason that the central bank allows the short rate to deviate further for larger volatilities is that it must weigh the cost of being far away from \( \theta \) with the cost of frequent intervention. For larger values of \( \sigma \), the cost of frequent intervention is dominant, and thus the central bank must allow the short rate more freedom before intervening. We can see in plot (a) of Figure 8 that the Vasicek model is allowed the most freedom for smaller values of \( r \), whereas the Black-Karasinski model is the most highly restricted. All three of these models have the same average volatility; however, for small values of \( r \), the Vasicek model has the most instantaneous volatility and the Black-Karasinski model has the least instantaneous volatility. This means that the potential for frequent intervention for small values of \( r \) is the most prevalent in the Vasicek model and therefore must be allowed the most freedom of the three models. This balance between running cost and frequent intervention, however, is exactly opposite for larger values of \( r \), as seen in plot (b) of Figure 8. Here the Black-Karasinski model is allowed the most freedom, and the Vasicek model is the most restricted. This is because for large values of \( r \) the Black-Karasinski model has the most instantaneous volatility, whereas the Vasicek model has the least.

In plots (c) and (d) of Figure 8, we show the size of the control issued by the central bank when the short rate process hits \( d \) or \( u \). We see that for larger values of \( \sigma \), the central bank issues a larger control for each model. This highlights the trade-off between fixed and proportional costs; in order to minimize costs when volatility is high, the central bank avoids paying the fixed cost frequently by issuing larger control and paying larger proportional costs. Furthermore, plot (c) shows that the size of control is the largest for the Vasicek model when the short rate hits \( d \) and smallest for the Black-Karasinski model. This again is related to the higher instantaneous volatility in the Vasicek model for smaller values for \( r \). As expected, the central bank issues the most control for the Black-Karasinski model when the short rate hits \( u \) because of its high local volatility for large values of \( r \) relative to the other two models.

In Figure 9 we plot the optimal value function for these three models when \( \sigma = 0.06 \). We see that the symmetry of the Vasicek model also holds for the value function, but again there is no symmetry for the CIR model, due to the asymmetry in volatility, or the Black-Karasinski model due to the asymmetry in both volatility and drift. We also see that the value function for the CIR and Black-Karasinski models are lower than that of the Vasicek model. Although

Figure 9. Value function for three models when \( \sigma = 0.06 \).

Notes. The gray dots represent the optimal \((d, D, U, u)\).
we set the parameters of these models so that the first two moments of the uncontrolled process match, we were not able to match higher moments. The CIR and Black-Karasinski models are both leptokurtic and therefore have fatter tails than the Vasicek model has. When the central bank imposes a control on the short rate process, it is effectively cutting off the tails of the long-term distribution. This has the most effect on the most leptokurtic distribution, which is the log-normal Black-Karasinski model, and therefore the value function for this model is also the lowest.

The examples we have presented here show the rich structure that comes from issuing control on interest rates. We have seen that this control can have drastic and interesting effects on the yield curve in many different ways. Beyond this the control behaves in rich and interesting ways for different interest rate models. Although this is a stylized version of a central bank’s intervention methods, these examples no doubt shed light on how to best control interest rates in different situations. Perhaps the most interesting insight is that for more volatile interest rates, the central bank should wait longer to intervene, and when it does finally issue a control, it should make a more drastic intervention.

6. Concluding Remarks

The focus of this paper was understanding the implications of interventions on the short rate by the Fed on the term structure of interest rates. However, to get to this we have had to develop and present a very general solution technique that can solve optimal control problems that have a fixed cost of control (thereby making it optimal to bring about discontinuities in the state dynamics). These problems, usually called impulse control problems, are notoriously hard to solve, and only numerical solutions have been available for special cases. Hence we hope that the method developed in this paper will be easily leveraged by researchers in various application areas like inventory management, portfolio selection, and healthcare.

Regardless of the federal government’s motivations to exert influence in the interest rate markets, whether it be to keep inflation in check or to maintain certain exchange rates, the influence should be exerted in the most rational way possible. This paper presented a model for the government’s ability to control the interest rate market, found the optimal control policy, and contrasted the resulting term structure with models that do not incorporate control. We also found that we are able to match the existing yield curves with our model by extracting the market price of risk from bond prices. This freedom provides for a rich model that can approximate the federal government’s goals and the free market’s response to those goals.

We have abstracted away from modeling the reasons that the federal government issues its control by simply assuming that the government has the ability to quantify its preferences and tolerances precisely. This, however, imposes a static nature to the government’s preferences. Allowing for a dynamic change in the governments preferences is more realistic but will require the careful modeling of the motivations for and side effects of intervening. This will undoubtedly yield a multidimensional impulse control model that is almost impossible to solve with what is available currently. Although this paper, for the first time, looks at the impulse control of these interest rates and the resulting term structures, going forward it is necessary to consider models that capture the complex relationships between the many macroeconomic variables and the central bank’s multifaceted objectives. These goals are important for the entire economy, and quantitatively assessing and optimally achieving these goals has the potential to lead to a stronger world market.

Supplemental Material

Supplemental material to this paper is available at http://dx.doi.org/10.1287/opre.2014.1270.

Endnote


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